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A relativisitc nonlocal theory of electromagnetic interactions with matter.

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A RELATIVISITC NONLOCAL THEORY OF ELECTROMAGNETIC INTERACTIONS WITH MATTER

by

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Abstract

A basic theory of nonlocal electromagnetic interactions with matter in the context of special relativistic continuum mechanics is studied. The relativistic balance laws are obtained. It is shown that local electromagnetic field and nonlocal electromagnetic field can be combined to yield a total electromagnetic field in such a way that the balance laws for the total electromagnetic fields become the classical balance laws. The nonlocal theory is based on the postulate that the total internal production of heat of the whole material world tube is non-negative. This postulate is shown to be equivalent to satisfaction of a specific functional inequality. The general solution of this functional inequality is obtained and it leads to the new forms of admissible constitutive relations. These constitutive relations include the well-known quasi-reversible solutions as special cases and, in addition, contain functional gradients of a dissipation potential with respect to convective time derivatives of the local and nonlocal state variables. These latter terms represent intrinsically dissipative structures of the material continuum. The general constitutive relations also contain terms that are nondissipative but dependent on convective time derivatives of the local and nonlocal state variables. Some of the implications of the various terms in the constitutive relations are obtained.

1. Introduction

This thesis studies the basic theory of nonlocal electromagnetic interactions with matter in the context of special relativistic continuum mechanics. Complete solutions of the global heat production inequality for electromagnetic interactions with matter are obtained and a fundamental decomposition of the field variables in terms of external and mutual or self fields is established.

It is well known that any theory involving electromagnetic fields should be based upon relativistic considerations. Within the scope of the special theory of relativity, the most natural and the simplest invariance structure is obtained under the Lorentz group of transformations. To this end, we adopt the notation that Latin indices take the values 1,2,3 and Greek indices the range 1,2,3,4 , with summation over the appropriate range of values in the case of a repeated index. Any exceptions to this rule are explicitly noted. The signs of the components of the fundamental tensor $\gamma^{\alpha\beta}$ are chosen so that the diagonal form is (1,1,1,-1) and not (-1,-1,-1,1) , i.e. $\gamma^{\alpha\beta}$ has the form

$$\gamma^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and hence $\det(\gamma^{\alpha\beta}) = -1$. For simplicity the speed of light c is set equal to unit, $c=1$ (cf. Grot and Eringen [1]). Rationalized natural units are employed throughout this thesis. Accordingly we take the dielectric constant of free space $\varepsilon_o=1$ and since $c=1$, the permeability of free space $\mu_o=1$. We also note that the world velocity vector u^α is defined by [1]

$$u^\alpha(x^\beta) \equiv \left\{ \frac{v^k}{\sqrt{1-v^2}} , \frac{1}{\sqrt{1-v^2}} \right\},$$

where

$$v^k \equiv \frac{\partial x^k(X^K,x^4)}{\partial x^4} , \quad v^2 \equiv \delta_{k\ell} v^k v^\ell , \quad u^\alpha u_\alpha = -1 ,$$

and the coordinates X^K and x^k are defined in detail in section 4 for $x^4 = ct$ $(c=1)$.

In section 2 a nonlocal theory of balance laws of a continuum description of electromagnetic interactions with matter from the relativistic point of view is established. It is shown that the local electromagnetic field and nonlocal electromagnetic field can be combined to yield a total electromagnetic field in such a way that the balance laws of electromagnetic fields become the classical balance laws. This shows that the balance laws of electromagnetic fields are linear in the local and the nonlocal electromagnetic fields.

In section 3 the second law of thermodynamics and a class of simple thermodynamic processes are introduced. The arguments for electromagnetic force and body couple in [1,2,3] are accepted, the extension to the nonlocal electromagnetic force and body couple are also posited. We simply assume that the local spin tensor gradient arises solely due to the occurrence of nonlocal body torques. The nonlocal theory presented in this section is based upon the postulate that the total internal production of heat of the whole material tube is non-negative. We note that the local and nonlocal electromagnetic force and body couple are nonlinear in the local and the nonlocal electromagnetic fields representatively, and hence we can not represent them in terms of the total electromagnetic fields.

In section 4 the functional form of the Helmholtz free energy is assumed and appropriate function spaces [4] are introduced. A fundamental functional inequality is then obtained which reduces to the global heat production inequality.

In section 5 a more general decomposition theorem for function spaces [4,5] is used. The general solution of the functional inequality is obtained and this leads to the new forms of admissible constitutive relations. The existence of a dissipation potential is established for material bodies in the presence of electromagnetic interactions with the new forms of admissible constitutive relations. These relations are intrinsically different from the quasi-reversible solutions available in the current literature. They include the quasi-reversible solutions as special cases and, in addition, contain functional gradients of a dissipation potential with respect to convective time derivatives of the local and nonlocal state variables. These latter terms represent intrinsically dissipative structures of the material continuum. The general constitutive relations also contain terms that are nondissipative but dependent on convective time derivatives of the local and nonlocal state variable. Some of the implications of the various terms in the constitutive relations are obtained.

Although the theory presented in this thesis is complete, there remain three underlying questions associated with the representations of specific functions and functionals. These questions are discussed in section 6 in that we point out some of the difficulties that will be encountered in their resolutions.

Definitive answers to these questions are not obtained since they lie outside of the scope of this thesis.

2. Relativistic Balance Laws

The local theories of balance laws in the continuum description of electromagnetic interactions with matter have been formulated from both the nonrelativistic and relativistic points of view, cf. Truesdell and Toupin [6], Dixon and Eringen [7], Grot and Eringen [1]. The nonlocal theories of balance laws have been obtained from the nonrelativistic point of view by Eringen [8]. We develop in this thesis a nonlocal theory of balance laws of a continuum description of electromagnetic interactions with matter from the relativistic point of view.

To simplify the analysis, we assume that the entire domain of electromagnetic interactions with matter has no surfaces of discontinuity or jump conditions, i.e., the entire domain of electromagnetic interactions with matter is continuous and homogeneous.

The integral form and differential equation form of the relativistically correct laws of balance, in the format of both global and local statements are given below: (cf. Grot and Eringen [1]).

(I) Conservation of Particle Number

The law of conservation of particle number assigns to every three-dimensional surface $S^{(3)}$, a positive scalar $N[S^{(3)}] = \int_{S^{(3)}} n^{\alpha} dS^{(3)}_{\alpha}$ that is conserved. By this we mean that

$$\oint n^{\alpha} dS^{(3)}_{\alpha} = 0 , \tag{2.1}$$

where

$$n^{\alpha} = n_o u^{\alpha} ,$$

$$n_o = -n^{\alpha} u_{\alpha} > 0 ,$$

and $\{u^{\alpha}\}$ = 4-velocity vector with $u^{\alpha}\gamma_{\alpha\beta}u^{\beta} \equiv u^{\alpha}u_{\alpha} = -1$, $((\gamma_{\alpha\beta})) = \mathrm{diag}(+1,+1,+1,-1)$. The scalar n_o is called the *rest frame particle number* and is related to the *particle number* n , by $n = \dfrac{n_o}{\sqrt{1-v^2}}$, where v^2 = square of particle 3-velocity and we have taken the speed of light in vacuum to be equal to unity. The use of the Green-Gauss theorem allows us to rewrite equation (2.1) in the equivalent form

$$\int_{(\beta)} (n_o u^{\alpha})_{,\alpha} \, dV^{(4)} = 0 , \tag{2.2}$$

where (β) is the world tube swept out by a given material body (B) in $V^{(4)}$ (space-time) as the proper time increases and $(\partial\beta)$ is the boundary of (β) .

Equation (2.2) can be localized [9] by introducing the *particle number residual* $\hat{n}$ (equivalence class) with

$$\int_{(\beta)} \hat{n}\, dV^{(4)} \equiv 0 \; . \qquad (2.3)$$

Combining (2.2) and (2.3) affords us the following localized (differential) equation

$$(n_o u^\alpha)_{,\alpha} = \hat{n} \; . \qquad (2.4)$$

We introduce the relativistic generalization of the material derivative D/Dt by

$$D = \frac{1}{\sqrt{1-v^2}} \frac{D}{Dt} \; .$$

Thus, if ϕ is a tensor under Lorentz transformations,

$$D\phi \equiv u^\alpha \phi_{,\alpha} \qquad (2.5)$$

is also a tensor under Lorentz transformations. Using (2.5), it is convenient to write (2.4) as

$$Dn_o + n_o u^\alpha_{,\alpha} = \hat{n} \; . \qquad (2.6)$$

If we integrate the local statement (2.6) over a subtube (p) of (β) we obtain

$$\int_{(p)} (Dn_o + n_o u^\alpha_{,\alpha}) dV^{(4)} = \int_{(p)} \hat{n}\, dV^{(4)}$$

$$= - \int_{(\beta)-(p)} \hat{n}\, dV^{(4)} \; , \qquad (2.7)$$

with the obvious interpretation of $\hat{n}$. From now on, we assume the system is locally closed with respect to particle number, i.e. $\hat{n} \equiv 0$ is assumed throughout our discussion.

(II) Balance of Energy-Momentum

The energy-momentum of a material body is determined by assigning to every three-dimensional subspace $S^{(3)}$, of a four-dimensional material tube (β), four function $P^{\mu}[S^{(3)}]$ of the form

$$P^{\mu}[S^{(3)}] = \int_{S^{(3)}} T^{\mu\nu} dS_{\nu}^{(3)} \; . \tag{2.8}$$

For every material tube (β) we then have the balance of energy-momentum in the following form:

$$\oint T^{\mu\nu} dS_{\nu}^{(3)} = \int_{(\beta)} f^{\mu} dV^{(4)} \; , \tag{2.9}$$

where

$$T^{\alpha\beta} = n_o \varepsilon u^{\alpha} u^{\beta} + u^{\alpha} q^{\beta} + p^{\alpha} u^{\beta} - t^{\alpha\beta} \; , \tag{2.10}$$

and

$$\begin{aligned} n_o \varepsilon &\equiv T^{\alpha\beta} u_{\alpha} u_{\beta} \; , \\ q^{\alpha} &\equiv -S^{\alpha}_{\beta} T^{\beta\nu} u_{\nu} \; , \\ p^{\alpha} &\equiv -S^{\alpha}_{\beta} T^{\beta\nu} u_{\nu} \; , \\ t^{\alpha\beta} &\equiv -S^{\alpha}_{\gamma} S^{\beta}_{\delta} T^{\gamma\delta} \; . \end{aligned} \tag{2.11}$$

Here S^{α}_{β} are the components of the space-like projector perpendicular to world velocity vector u^{β} :

$$S^{\alpha}_{\beta} \equiv \delta^{\alpha}_{\beta} + u^{\alpha}u_{\beta} \ ,$$
$$S^{\alpha}_{\beta}u^{\beta} = S^{\beta}_{\alpha}u_{\beta} = 0 \ . \tag{2.12}$$

The meaning of the various induced quantities are

$n_o \varepsilon u^{\alpha}u^{\beta} \equiv$ kinetic energy-momentum,

$q^{\beta} \equiv$ heat flow four-vector,

$p^{\alpha} \equiv$ nonmechanical momentum four-vector,

$t^{\alpha\beta} \equiv$ relativistic stress tensor,

$f^{\mu} \equiv$ body force and energy supply per unit volume,

so that $(2.11)_2$, $(2.11)_3$ and $(2.11)_4$ yield

$$q^{\alpha}u_{\alpha} = 0 \ ,$$
$$p^{\alpha}u_{\alpha} = 0 \ , \tag{2.13}$$
$$t^{\alpha\beta}u_{\beta} = t^{\beta\alpha}u_{\beta} = 0 \ .$$

The Green-Gauss theorem and (2.9) thus yield the global statement

$$\int_{(\beta)} (T^{\mu\nu}{}_{,\nu} - f^{\mu})dV^{(4)} = 0 \ . \tag{2.14}$$

This statement can be localized by introducing the *force-energy residual* $\hat{f}^{\mu}$ (equivalence class) with the property

$$\int_{(\beta)} n_o \hat{f}^{\mu} dV^{(4)} \equiv 0 \ . \tag{2.15}$$

The four-vector $n_o\hat{f}^\mu$ can be decomposed into a time-like component and space-like components by

$$n_o\hat{f}^\alpha = n_o\hat{f}_o u^\alpha + n_o\hat{f}^\alpha_{(3)} \ ,$$

where

$$\begin{aligned} n_o\hat{f}_o &\equiv -n_o\hat{f}^\alpha u_\alpha \equiv \text{internal energy supply} \ , \\ n_o\hat{f}^\alpha_{(3)} &\equiv S^\alpha_\beta n_o\hat{f}^\beta \equiv \text{internal body force} \ , \\ n_o\hat{f}^\alpha_{(3)} u_\alpha &= 0 \ . \end{aligned} \tag{2.16}$$

Combining (2.14) and (2.15), we are led in the usual way to the following local (differential) equations

$$T^{\mu\nu}{}_{,\nu} - f^\mu = n_o\hat{f}^\mu \ . \tag{2.17}$$

If we integrate the local statement (2.17) over a subtube (p) of (β) we have the local statement

$$\begin{aligned} \int_{(p)} (T^{\mu\nu}{}_{,\nu} - f^\mu) dV^{(4)} &= \int_{(p)} n_o\hat{f}^\mu dV^{(4)} \\ &= -\int_{(\beta)-(p)} n_o\hat{f}^\mu dV^{(4)} \ , \end{aligned} \tag{2.18}$$

with the obvious interpretation of $n_o\hat{f}^\mu$. Substituting (2.10) into (2.17), multiplying the results by u_α and S^γ_α , and using, (2.5), (2.12), (2.13) and (2.16), we obtain the equations

$$n_o D\varepsilon + q^\beta{}_{,\beta} + P^\beta D u_\beta - t^{\alpha\beta} u_{\alpha,\beta} + f^\alpha u_\alpha = n_o\hat{f}_o \ , \tag{2.19}$$

$$n_o \varepsilon D u^\alpha + n_o S^\alpha_\gamma D\left(\frac{p^\gamma}{n_o}\right) + q^\beta u^\alpha_{,\beta} - t^{\alpha\beta}_{,\beta} + t^{\beta\nu} u_{\beta,\nu} u^\alpha$$

$$- S^\alpha_\beta f^\beta = n_o \hat{f}^\alpha_{(3)} . \qquad (2.20)$$

We note that only three of the four equations (2.20) are independent.

(III) Balance of Moment of Energy-Momentum

We assign to every three-dimensional subspace $S^{(3)}$, of the four-dimensional material tube (β), a skew-symmetric tensor function $M^{\alpha\beta}[S^{(3)}]$ by the relation

$$M^{\alpha\beta}[S^{(3)}] = \int_{S^{(3)}} M^{\alpha\beta\mu} dS^{(3)}_\mu .$$

The global law of balance of moment of energy-momentum then takes the following form

$$\oint M^{\alpha\beta\mu} dS^{(3)}_\mu = \int_{(\beta)} \{x^{[\alpha}f^{\beta]} + L^{\alpha\beta}\} dV^{(4)} , \qquad (2.21)$$

where

$$\begin{aligned} M^{\alpha\beta\mu} &\equiv x^{[\alpha}T^{\beta]\mu} + S^{\alpha\beta\mu} , \\ M^{(\alpha\beta)\mu} &= 0 , \\ S^{(\alpha\beta)\mu} &= 0 , \\ L^{(\alpha\beta)} &= 0 , \end{aligned} \qquad (2.22)$$

and

$S^{\alpha\beta\mu} \equiv$ spin tensor ,

$L^{\alpha\beta} \equiv$ four-dimensional analogue of the body toque .

If we use the Green-Gauss theorem, equation (2.21) becomes

$$\int_{(\beta)} (M^{\alpha\beta\mu}{}_{,\mu} - x^{[\alpha}f^{\beta]} - L^{\alpha\beta})\, dV^{(4)} = 0 \ . \qquad (2.23)$$

This statement can be localized by introducing the *body toque residual* $\hat{\ell}^{\alpha\beta}$ (equivalence class) such that

$$\int_{(\beta)} \hat{\ell}^{\alpha\beta} dV^{(4)} \equiv 0 \ , \qquad \hat{\ell}^{(\alpha\beta)} \equiv 0 \ . \qquad (2.24)$$

Now, it is possible to decompose $\hat{\ell}^{\alpha\beta}$ into

$$\hat{\ell}^{\alpha\beta} = \hat{\ell}^{\beta}_{o} u^{\alpha} - \hat{\ell}^{\alpha}_{o} u^{\beta} + \hat{\hat{\ell}}^{\alpha\beta} \ . \qquad (2.25)$$

Since $\hat{\ell}^{(\alpha\beta)} = 0$ implies $\hat{\hat{\ell}}^{(\alpha\beta)} = 0$, $\hat{\ell}^{\beta}_{o} \equiv \hat{\ell}^{\alpha\beta} u_{\alpha}$ is called the *nonlocal spin vector* and $\hat{\hat{\ell}}^{\alpha\beta}$ is called *nonlocal spin tensor*. A combination of (2.23) and (2.24) leads in the usual way to the following local (differential) equations

$$M^{\alpha\beta\mu}{}_{,\mu} - x^{[\alpha}f^{\beta]} - L^{\alpha\beta} = \hat{\ell}^{\alpha\beta} \ . \qquad (2.26)$$

By use of equation (2.17) and $(2.22)_1$, we can replace equation (2.26) by

$$S^{\alpha\beta\mu}{}_{,\mu} - T^{[\alpha\beta]} - L^{\alpha\beta} - n_o x^{[\alpha}\hat{f}^{\beta]} = \hat{\ell}^{\alpha\beta} \ . \qquad (2.27)$$

If we integrate the local statement (2.27) over a subtube (p) of (β) we have the local statement

$$\int_{(p)} [S^{\alpha\beta\mu}{}_{,\mu} - T^{[\alpha\beta]} - L^{\alpha\beta} - n_o x^{[\alpha}\hat{f}^{\beta]}] dV^{(4)} = \int_{(p)} \hat{\ell}^{\alpha\beta} dV^{(4)}$$

$$= - \int_{(\beta)-(p)} \hat{\ell}^{\alpha\beta} dV^{(4)} , \qquad (2.28)$$

and the interpretation of $\hat{\ell}^{\alpha\beta}$ follows.

(IV) Conservation of Charge

The law of conservation of charge assigns to every three-dimensional subspace $S^{(3)}$ of a four-dimensional material tube (β) a scalar function $Q[S^{(3)}]$ of the form

$$Q[S^{(3)}] = \int_{S^{(3)}} \sigma^{\alpha} dS_{\alpha}^{(3)}$$

with the property that

$$\oint \sigma^{\alpha} dS_{\alpha}^{(3)} = 0 , \qquad (2.29)$$

for every three-dimensional circuit. Here, σ^{α} is called the *local charge-current vector* and

$$\sigma^{\alpha} = n_o \sigma_o u^{\alpha} + j^{\alpha}$$

where

$$n_o \sigma_o \equiv -\sigma^{\alpha} u_{\alpha} ,$$

$$j^{\alpha} \equiv S_{\alpha}^{\beta} \sigma^{\alpha} , \quad j^{\alpha} u_{\alpha} = 0 . \qquad (2.30)$$

Thus $n_o \sigma_o$ is the *local charge density* and j^{α} the *local conduction current.*

The use of the Green-Gauss theorem allows us to rewrite equation (2.29) in the equivalent form

$$\int_{(\beta)} \sigma^{\alpha}_{,\alpha} dV^{(4)} = 0 \ . \tag{2.31}$$

This equation can be localized by introducing the localization residual $\hat{\sigma}$ (equivalence class) such that

$$\int_{(\beta)} \hat{\sigma} \ dV^{(4)} \equiv 0 \ . \tag{2.32}$$

Combining (2.31) and (2.32), we are led in the usual way to the following local (differential) equation

$$\sigma^{\alpha}_{,\alpha} = \hat{\sigma} \ . \tag{2.33}$$

Recalling (2.5) and $(2.30)_1$, we find it convenient to write (2.33) as

$$n_o D\sigma_o + j^{\alpha}_{,\alpha} = \hat{\sigma} \ . \tag{2.34}$$

We now assume that

$$\hat{\sigma} = -\hat{\sigma}^{\alpha}_{,\alpha} \tag{2.35}$$

where $\hat{\sigma}^{\alpha}$ is called the *nonlocal charge-current vector,*

$$\hat{\sigma}^{\alpha} = n_o \hat{\sigma}_o u^{\alpha} + \hat{j}^{\alpha} \ ,$$

and

$$n_o \hat{\sigma}_o \equiv -\hat{\sigma}^{\alpha} u_{\alpha} \ ,$$

$$\hat{j}^{\alpha} \equiv S^{\alpha}_{\beta} \sigma^{\beta} \ , \qquad \hat{j}^{\alpha} u_{\alpha} = 0 \ . \tag{2.36}$$

Thus, $n_o \hat{\sigma}_o$ is the *nonlocal charge density* and $\hat{j}^{\alpha}$ the *nonlocal conduction current.* Using (2.5) and

(2.36), we see that (2.35) becomes

$$\hat{\sigma} = -n_o D\hat{\sigma}_o - \hat{j}^{\alpha}_{;\alpha} . \tag{2.37}$$

Thus, use of (2.37) shows that (2.34) becomes

$$n_o D(\sigma_o+\hat{\sigma}_o) + (j^{\alpha}+\hat{j}^{\alpha})_{,\alpha} = 0 . \tag{2.38}$$

Equation (2.38) is the expression of the resulting local law of conservation of charge.

If we integrate the local statement (2.38) over a subtube (p) of (β) we have the local statement

$$\int_{(p)} (n_o D\sigma_o + j^{\alpha}_{;\alpha})dV^{(4)} = -\int_{(p)} (n_o D\hat{\sigma}_o + \hat{j}^{\alpha}_{;\alpha})dV^{(4)}$$
$$= \int_{(\beta)-(p)} (n_o D\hat{\sigma}_o + \hat{j}^{\alpha}_{;\alpha})dV^{(4)} . \tag{2.39}$$

Since

$$\int_{(\beta)} \hat{\sigma}\ dV^{(4)} \equiv 0 ,$$

we see that

$$\int_{(\beta)} \hat{\sigma}^{\alpha}_{;\alpha} dV^{(4)} \equiv 0 ,$$

and use of the Green-Gauss theorem gives us

$$\int_{S^{(3)}} \hat{\sigma}^{\alpha} dS_{\alpha} \equiv 0 , \tag{2.40}$$

for fixed $S^{(3)} \subset (\partial\beta)$.

(V) Conservation of Magnetic Flux

The conservation of magnetic flux is obtained by assigning to every two-dimensional subspace $S_{(2)}$ in space-time a scalar quantity $\Phi[S_{(2)}]$, called the magnetic flux,

$$\Phi[S_{(2)}] = \frac{1}{2}\int_{S_{(2)}} \phi_{\alpha\beta} dS^{\alpha\beta}_{(2)} \ , \qquad \phi_{\alpha\beta} = -\phi_{\beta\alpha} \ ,$$

with the property that it vanishes for every two-dimensional circuit

$$\oint \phi_{\alpha\beta} dS^{\alpha\beta}_{(2)} = 0 \ . \tag{2.41}$$

Stoke's theorem shows that equation (2.41) is equivalent to

$$\int_{S^{(3)}} \varepsilon^{\alpha\beta\gamma\delta}\phi_{\gamma\delta,\beta} dS^{(3)}_{\alpha} = 0 \ , \tag{2.42}$$

for fixed $S^{(3)} \subset (\partial\beta)$ where $\varepsilon^{\alpha\beta\gamma\delta} \equiv e_{\alpha\beta\gamma\delta}$ = permutation symbol. We also define $\varepsilon^{\alpha\beta\gamma\delta} \equiv -e^{\alpha\beta\gamma\delta}$, and, since $\det(\gamma^{\mu\nu}) = -1$, we obtain

$$\varepsilon^{\alpha\beta\gamma\delta} = \gamma^{\alpha\alpha_1}\gamma^{\beta\beta_1}\gamma^{\gamma\gamma_1}\gamma^{\delta\delta_1}\varepsilon_{\alpha_1\beta_1\gamma_1\delta_1} \ . \tag{2.43}$$

The *local magnetic flux tensor* $\phi_{\alpha\beta}$ has physical components

$$\phi_{\alpha\beta} = [\text{dual } \underset{\sim}{B}, \underset{\sim}{E}] \ , \tag{2.44}$$

where $\underset{\sim}{B}$ is the local density of magnetic flux and $\underset{\sim}{E}$ is the local electric field. We defined $\bar{\phi}^{\alpha\beta}$ by

$$\bar{\phi}^{\alpha\beta} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\phi_{\gamma\delta} \ . \tag{2.45}$$

In terms of $\underset{\sim}{E}$ and $\underset{\sim}{B}$, $\bar{\phi}$ is

$$\bar{\phi}^{\alpha\beta} = [\text{dual } \underset{\sim}{E}, \underset{\sim}{B}] \, . \tag{2.46}$$

It is possible to decompose $\phi_{\alpha\beta}$ into

$$\phi_{\alpha\beta} = \mathcal{E}_\beta u_\alpha - \mathcal{E}_\alpha u_\beta + \overset{*}{\phi}_{\alpha\beta} \, , \tag{2.47}$$

where

$$\mathcal{E}_\alpha \equiv \phi_{\alpha\beta} u^\beta \, , \qquad \mathcal{E}_\alpha u^\alpha = 0 \, ,$$

$$\overset{*}{\phi}_{\alpha\beta} \equiv S^\gamma_\alpha S^\delta_\beta \phi_{\gamma\delta} \, , \qquad \overset{*}{\phi}_{\alpha\beta} u^\beta = 0 \, . \tag{2.48}$$

It is also convenient to define the field $\overset{*}{\phi}_{\alpha\beta}$ by

$$\overset{*}{\phi}_{\alpha\beta} \equiv \varepsilon_{\alpha\beta\gamma\delta} \mathcal{B}^\gamma u^\delta \, , \qquad \mathcal{B}^\alpha u_\alpha = 0 \, . \tag{2.49}$$

Equation (2.47) thus becomes

$$\phi_{\alpha\beta} = \mathcal{E}_\beta u_\alpha - \mathcal{E}_\alpha u_\beta + \varepsilon_{\alpha\beta\gamma\delta} \mathcal{B}^\gamma u^\delta \, . \tag{2.50}$$

Similarly, $\bar{\phi}^{\alpha\beta}$ can be decomposed into

$$\bar{\phi}^{\alpha\beta} = \mathcal{B}^\alpha u^\beta - \mathcal{B}^\beta u^\alpha + \varepsilon^{\alpha\beta\gamma\delta} \mathcal{E}_\gamma u_\delta \, . \tag{2.51}$$

The two decompositions (2.50) and (2.51) are found to be useful in the formulation of constitutive equations for the local electromagnetic quantities and for expressing the interactions of local electromagnetic fields with matter.

The four-vectors $\mathcal{E}^\alpha$ and $\mathcal{B}^\alpha$ are given in terms of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ by

$$\mathcal{E}^{\alpha} = \left[\frac{\underset{\sim}{E}+\underset{\sim}{V}\times\underset{\sim}{B}}{\sqrt{1-v^2}}, \frac{\underset{\sim}{V}\cdot\underset{\sim}{E}}{\sqrt{1-v^2}}\right], \quad \mathcal{B}^{\alpha} = \left[\frac{\underset{\sim}{B}-\underset{\sim}{V}\times\underset{\sim}{E}}{\sqrt{1-v^2}}, \frac{\underset{\sim}{V}\cdot\underset{\sim}{B}}{\sqrt{1-v^2}}\right]. \qquad (2.52)$$

We can localize the equation (2.42) by introducing the localization residual $\hat{\Phi}^{\alpha}$ (equivalence class) with

$$\int_{S^{(3)}} \hat{\Phi}^{\alpha} dS_{\alpha}^{(3)} \equiv 0 , \qquad (2.53)$$

for fixed $S^{(3)} \subset (\partial\beta)$. Combining (2.42) and (2.53) we are led in the usual way to the following local (differential) equations

$$\varepsilon^{\alpha\beta\gamma\delta}\phi_{\gamma\delta,\beta} = \hat{\Phi}^{\alpha} . \qquad (2.54)$$

Now, defined $\hat{\Phi}^{\alpha}$ by

$$\hat{\Phi}^{\alpha} = -\varepsilon^{\alpha\beta\gamma\delta}\hat{\phi}_{\gamma\delta,\beta} . \qquad (2.55)$$

The *nonlocal magnetic flux tensor* $\hat{\phi}_{\alpha\beta}$ is given in terms of physical components by

$$\hat{\phi}_{\alpha\beta} = [\text{dual } \hat{\underset{\sim}{B}}, \hat{\underset{\sim}{E}}] , \qquad (2.56)$$

where $\hat{\underset{\sim}{B}}$ is the *nonlocal density of magnetic flux* and $\hat{\underset{\sim}{E}}$ is the *nonlocal electric field.* We define $\bar{\hat{\phi}}^{\alpha\beta}$ by

$$\bar{\hat{\phi}}^{\alpha\beta} = -\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\hat{\phi}_{\gamma\delta} . \qquad (2.57)$$

In terms of $\hat{\underset{\sim}{E}}$ and $\hat{\underset{\sim}{B}}$, $\bar{\hat{\phi}}$ is

$$\bar{\hat{\phi}}^{\alpha\beta} = [\text{dual } \hat{\underset{\sim}{E}}, \hat{\underset{\sim}{B}}] . \qquad (2.58)$$

Using the same method as that used in deriving the equations (2.50) and (2.51), we can decompose $\hat{\phi}_{\alpha\beta}$ and $\bar{\hat{\phi}}^{\alpha\beta}$ into

$$\hat{\phi}_{\alpha\beta} = \hat{\mathcal{E}}_\beta u_\alpha - \hat{\mathcal{E}}_\alpha u_\beta + \varepsilon_{\alpha\beta\gamma\delta}\hat{\mathcal{B}}^\gamma u^\delta \ , \tag{2.59}$$

$$\bar{\hat{\phi}}^{\alpha\beta} = \hat{\mathcal{B}}^\alpha u^\beta - \hat{\mathcal{B}}^\beta u^\alpha + \varepsilon^{\alpha\beta\gamma\delta}\hat{\mathcal{E}}_\gamma u_\delta \ , \tag{2.60}$$

where

$$\hat{\mathcal{E}}^\alpha = \left[\frac{\hat{\underset{\sim}{E}}+\underset{\sim}{V}\times\hat{\underset{\sim}{B}}}{\sqrt{1-v^2}} \ , \ \frac{\underset{\sim}{V}\cdot\hat{\underset{\sim}{E}}}{\sqrt{1-v^2}}\right],$$

$$\hat{\mathcal{B}}^\alpha = \left[\frac{\hat{\underset{\sim}{B}}-\underset{\sim}{V}\times\hat{\underset{\sim}{E}}}{\sqrt{1-v^2}} \quad \frac{\underset{\sim}{V}\cdot\hat{\underset{\sim}{B}}}{\sqrt{1-v^2}}\right]. \tag{2.61}$$

The two decompositions (2.59) and (2.60) are useful in the formulation of constitutive equations for the nonlocal electromagnetic quantities and for expressing the interaction of nonlocal electromagnetic fields with matter.

Using (2.45), (2.55) and (2.57), we see that equation (2.54) becomes

$$\bar{\phi}^{\alpha\beta}{}_{,\beta} = -\bar{\hat{\phi}}^{\alpha\beta}{}_{,\beta} \ . \tag{2.62}$$

Substituting (2.51) and (2.60) into (2.62), multiplying the results by u_α and S^γ_α, and using the definition of (2.5), the properties (2.48), (2.49) and the short hand notation $\overset{*}{\varepsilon}{}^{\alpha\beta\gamma} \equiv \varepsilon^{\alpha\beta\gamma\delta}u_\delta$, we obtain

$$S^\gamma_\beta \mathcal{B}^\beta_{,\gamma} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\mathcal{E}_\alpha u_{\beta,\gamma} = -(S^\gamma_\beta \hat{\mathcal{B}}^\beta_{,\gamma} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\hat{\mathcal{E}}_\alpha u_{\beta,\gamma}) \ , \tag{2.63}$$

and

$$\overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\mathcal{E}_{\gamma,\beta} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\mathcal{E}_{\beta}Du_{\gamma} + S^{\alpha}_{\beta}D\mathcal{B}^{\beta} + \mathcal{B}^{\alpha}u^{\beta}_{;\beta} - \mathcal{B}^{\beta}u^{\alpha}_{;\beta}$$

$$= -\left[\overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\hat{\mathcal{E}}_{\gamma,\beta} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\hat{\mathcal{E}}_{\beta}Du_{\gamma} + S^{\alpha}_{\beta}D\hat{\mathcal{B}}^{\beta} + \hat{\mathcal{B}}^{\alpha}u^{\beta}_{;\beta} - \hat{\mathcal{B}}^{\beta}u^{\alpha}_{;\beta}\right] . \quad (2.64)$$

If we rewrite (2.63) and (2.64), we obtain

$$S^{\gamma}_{\beta}(\mathcal{B}^{\beta}+\hat{\mathcal{B}}^{\beta})_{,\gamma} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}(\mathcal{E}_{\alpha}+\hat{\mathcal{E}}_{\alpha})u_{\beta,\gamma} = 0 , \quad (2.65)$$

and

$$\overset{*}{\varepsilon}{}^{\alpha\beta\gamma}(\mathcal{E}_{\gamma}+\hat{\mathcal{E}}_{\gamma})_{,\beta} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}(\mathcal{E}_{\beta}+\hat{\mathcal{E}}_{\beta})Du_{\gamma} + S^{\alpha}_{\beta}D(\mathcal{B}^{\beta}+\hat{\mathcal{B}}^{\beta})$$

$$+ (\mathcal{B}^{\alpha}+\hat{\mathcal{B}}^{\alpha})u^{\beta}_{;\beta} - (\mathcal{B}^{\beta}+\hat{\mathcal{B}}^{\beta})u^{\alpha}_{;\beta} = 0 . \quad (2.66)$$

We specifically note that equation (2.65) is equivalent to magnetic flux equation, and that only three of equation (2.66) are independent. These three independent equations are equivalent to Faraday's Law. From equation (2.53) and (2.55), we note that

$$\int_{S^{(3)}}\left(-\varepsilon^{\alpha\beta\gamma\delta}\hat{\phi}_{\gamma\delta,\beta}\right)dS^{(3)}_{\alpha} \equiv 0 ,$$

for fixed $S^{(3)} \subset (\partial\beta)$. This implies

$$\oint \hat{\phi}_{\alpha\beta}dS^{\alpha\beta}_{(2)} \equiv 0 , \quad (2.67)$$

for every fixed two-dimensional circuit contained in $S^{(3)}$.

(VI) Ampere's and Guass' Laws

Ampere's and Gauss' laws are combined into one Lorentz invariant law by assigning to every two-dimensional subspace $S_{(2)}$ a scalar invariant $\Gamma[S_{(2)}]$

$$\Gamma[S_{(2)}] = \int_{S_{(2)}} g^{\alpha\beta} d\bar{S}^{(2)}_{\alpha\beta} ,$$

where

$$g^{\alpha\beta} = -g^{\beta\alpha} ,$$
$$d\bar{S}^{(2)}_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} dS^{\gamma\delta}_{(2)} . \tag{2.68}$$

For every closed two-dimensional circuit enclosing a three-dimensional subspace $S^{(3)} \subset (\partial\beta)$, Ampere's and Gauss' law are then expressed by

$$\oint g^{\alpha\beta} d\bar{S}^{(2)}_{\alpha\beta} = \int_{S^{(3)}} \sigma^{\alpha} dS^{(3)}_{\alpha} . \tag{2.69}$$

Stoke's theorem shows that equation (2.69) is equivalent to

$$\int_{S^{(3)}} (g^{\alpha\beta}{}_{,\beta} - \sigma^{\alpha}) dS^{(3)}_{\alpha} = 0 , \tag{2.70}$$

for fixed $S^{(3)} \subset (\partial\beta)$, where the (local) charge-current vector σ^{α} is defined by (2.30). The *local electric displacement-magnetic field intensity tensor* $g^{\alpha\beta}$ is given in terms of physical quantities by

$$g^{\alpha\beta} = [\text{dual } \underset{\sim}{H}, -\underset{\sim}{D}] , \tag{2.71}$$

where $\underset{\sim}{D}$ is the *local electric displacement* and $\underset{\sim}{H}$ is the *local magnetic field intensity.*

We can decompose $g^{\alpha\beta}$ in a fashion similar to (2.50) to obtain

$$g^{\alpha\beta} = \mathcal{D}^{\beta}u^{\alpha} - \mathcal{D}^{\alpha}u^{\beta} + \varepsilon^{\alpha\beta\gamma\delta}\mathcal{H}_{\gamma}u_{\delta} , \tag{2.72}$$

where

$$\mathcal{D}^{\beta} \equiv g^{\alpha\beta}u_{\alpha} , \quad \mathcal{D}^{\beta}u_{\beta} = 0 ,$$

$$\mathcal{H}_{\alpha} = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}g^{\beta\gamma}u^{\delta} , \quad \mathcal{H}_{\alpha}u^{\alpha} = 0 . \tag{2.73}$$

The decomposition (2.72) is also found to be useful in the formulation of constitutive equations for the local electromagnetic quantities and for expressing the interactions of local electromagnetic fields with matter. In terms of the local electromagnetic fields $\underset{\sim}{D}$ and $\underset{\sim}{H}$, $\mathcal{D}^{\alpha}$ and $\mathcal{H}^{\alpha}$ are

$$\mathcal{D}^{\alpha} = \left[\frac{\underset{\sim}{D}+\underset{\sim}{V}\times\underset{\sim}{H}}{\sqrt{1-v^2}} , \frac{\underset{\sim}{V}\cdot\underset{\sim}{D}}{\sqrt{1-v^2}}\right] ,$$

$$\mathcal{H}^{\alpha} = \left[\frac{\underset{\sim}{H}-\underset{\sim}{V}\times\underset{\sim}{D}}{\sqrt{1-v^2}} , \frac{\underset{\sim}{V}\cdot\underset{\sim}{H}}{\sqrt{1-v^2}}\right] . \tag{2.74}$$

We can localize the equation (2.70) by introducing the localization residuals $\hat{G}^{\alpha}$ (equivalence class) with

$$\int_{S^{(3)}} \hat{G}^{\alpha}dS^{(3)}_{\alpha} \equiv 0 , \tag{2.75}$$

for fixed $S^{(3)} \subset (\partial B)$. A combination of (2.70) and (2.75) leads in the usual way to the following local (differential) equations

$$g^{\alpha\beta}{}_{,\beta} - \sigma^{\alpha} = \hat{G}^{\alpha} . \tag{2.76}$$

Now, defined $\hat{G}^{\alpha}$ by

$$\hat{G}^{\alpha} = -(\hat{\underset{\sim}{g}}^{\alpha\beta}{}_{,\beta} - \hat{\sigma}^{\alpha}) , \tag{2.77}$$

where the nonlocal charge-current vector $\hat{\sigma}^{\alpha}$ is defined by equation (2.36) and (2.40) so that

$$\int_{S^{(3)}} \hat{\sigma}^{\alpha} dS_{\alpha}^{(3)} \equiv 0 ,$$

for fixed $S^{(3)} \subset (\partial\beta)$. The *nonlocal electric displacement-magnetic field intensity tensor* $\hat{g}^{\alpha\beta}$ is given in terms of physical components by

$$\hat{g}^{\alpha\beta} = [\text{dual } \hat{\underset{\sim}{H}}, -\hat{\underset{\sim}{D}}] , \tag{2.78}$$

where $\hat{\underset{\sim}{D}}$ is the *nonlocal electric displacement* and $\hat{\underset{\sim}{H}}$ is the *nonlocal magnetic field intensity.*
Similar to (2.59), we can decompose $\hat{g}^{\alpha\beta}$ into

$$\hat{g}^{\alpha\beta} = \hat{\mathcal{D}}^{\beta} u^{\alpha} - \hat{\mathcal{D}}^{\alpha} u^{\beta} + \varepsilon^{\alpha\beta\gamma\delta} \hat{\mathcal{H}}_{\gamma} u_{\delta} , \tag{2.79}$$

where

$$\hat{\mathcal{D}}^{\beta} \equiv \hat{g}^{\alpha\beta} u_{\alpha} , \quad \hat{\mathcal{D}}^{\beta} u_{\beta} = 0 ,$$

$$\hat{\mathcal{H}}_{\gamma} \equiv \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} \hat{g}^{\beta\gamma} u^{\delta} , \quad \hat{\mathcal{H}}_{\alpha} u^{\alpha} = 0 . \tag{2.80}$$

In terms of the nonlocal electromagnetic fields $\hat{\underset{\sim}{D}}$ and $\hat{\underset{\sim}{H}}$, $\hat{\mathcal{D}}^{\alpha}$ and $\hat{\mathcal{H}}^{\alpha}$ are:

$$\hat{\mathcal{D}}^{\alpha} = \left[\frac{\hat{\underset{\sim}{D}} + \underset{\sim}{V} \times \hat{\underset{\sim}{H}}}{\sqrt{1-v^2}} , \frac{\underset{\sim}{V} \cdot \hat{\underset{\sim}{D}}}{\sqrt{1-v^2}} \right] ,$$

$$\hat{\mathcal{H}}^{\alpha} = \left[\frac{\hat{\underset{\sim}{H}} - \underset{\sim}{V} \times \hat{\underset{\sim}{D}}}{\sqrt{1-v^2}} \; , \; \frac{\underset{\sim}{V} \cdot \hat{\underset{\sim}{H}}}{\sqrt{1-v^2}} \right] . \tag{2.81}$$

From (2.75) and (2.40), we also obtain

$$\int_{S^{(3)}} \hat{g}^{\alpha\beta}{}_{,\beta} dS^{(3)}_{\alpha} \equiv 0 \; , \tag{2.82}$$

for fixed $S^{(3)} \subset (\partial\beta)$. This implies

$$\oint \hat{g}^{\alpha\beta} d\bar{S}^{(2)}_{\alpha\beta} \equiv 0 \; , \tag{2.83}$$

for every fixed two-dimensional circuit contained in $S^{(3)}$. By use of equation (2.77), we see that (2.76) becomes

$$g^{\alpha\beta}{}_{,\beta} - \sigma^{\alpha} = -(\hat{g}^{\alpha\beta}{}_{,\beta} - \hat{\sigma}^{\alpha}) \; . \tag{2.84}$$

Substituting (2.30), (2.36), (2.72) and (2.79) into (2.84), multiplying the results by u_{α} and S^{γ}_{α} , using the definition (2.5), and properties (2.73) and (2.80), we obtain

$$S^{\gamma}_{\beta}\mathcal{D}^{\beta}{}_{,\gamma} + \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\mathcal{H}_{\alpha}u_{\beta,\gamma} - n_o\sigma_o = -\left[S^{\gamma}_{\beta}\hat{\mathcal{D}}^{\beta}{}_{,\gamma} + \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\hat{\mathcal{H}}_{\alpha}u_{\beta,\gamma} - n_o\hat{\sigma}_o\right] , \tag{2.85}$$

and

$$\overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\mathcal{H}_{\gamma,\beta} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\mathcal{H}_{\beta}Du_{\gamma} - S^{\alpha}_{\beta}D\mathcal{D}^{\beta} + \mathcal{D}^{\beta}u^{\alpha}{}_{,\beta} - \mathcal{D}^{\alpha}u^{\beta}{}_{,\beta} - j^{\alpha}$$

$$= -\left[\overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\hat{\mathcal{H}}_{\gamma,\beta} - \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}\hat{\mathcal{H}}_{\beta}Du_{\gamma} - S^{\alpha}_{\beta}D\hat{\mathcal{D}}^{\beta} + \hat{\mathcal{D}}^{\beta}u^{\alpha}{}_{,\beta} - \hat{\mathcal{D}}^{\alpha}u^{\beta}{}_{,\beta} - \hat{j}^{\alpha}\right] . \tag{2.86}$$

These can be rewritten in the equivalent forms

$$S^{\gamma}_{\beta}(\mathcal{D}^{\beta}+\hat{\mathcal{D}}^{\beta})_{,\gamma} + \overset{*}{\varepsilon}{}^{\alpha\beta\gamma}(\mathcal{H}_{\alpha}+\hat{\mathcal{H}}_{\alpha})u_{\beta,\gamma} - n_o(\sigma_o+\hat{\sigma}_o) = 0 \; , \tag{2.87}$$

and

$$\overset{*}{\varepsilon}^{\alpha\beta\gamma}(\mathcal{H}_\gamma+\hat{\mathcal{H}}_\gamma)_{,\beta} - \overset{*}{\varepsilon}^{\alpha\beta\gamma}(\mathcal{H}_\beta+\hat{\mathcal{H}}_\beta)Du_\gamma - S^\alpha_\beta D(\mathcal{D}^\beta+\hat{\mathcal{D}}^\beta)$$

$$+ (\mathcal{D}^\beta+\hat{\mathcal{D}}^\beta)u^\alpha_{,\beta} - (\mathcal{D}^\alpha+\hat{\mathcal{D}}^\alpha)u^\beta_{,\beta} - (j^\alpha+\hat{j}^\alpha) = 0 \ . \qquad (2.88)$$

We note that equation (2.87) is equivalent to Gauss' Law, and only three of equations (2.88) are independent. The three independent equations in the system (2.88) are equivalent to Ampere's Law.

At this point, we can draw a very important conclusion concerning the presence of local and nonlocal electromagnetic field quantities. We define $\sigma^\alpha_{(total)}$, $\underset{\sim}{E}_{total}$, $\underset{\sim}{B}_{total}$, $\underset{\sim}{H}_{total}$ and $\underset{\sim}{D}_{total}$ by

$$\sigma^\alpha_{(total)} = \sigma^\alpha + \hat{\sigma}^\alpha \ , \qquad (2.89)$$

$$\underset{\sim}{E}_{total} = \underset{\sim}{E} + \hat{\underset{\sim}{E}} \ , \qquad (2.90)$$

$$\underset{\sim}{B}_{total} = \underset{\sim}{B} + \hat{\underset{\sim}{B}} \ , \qquad (2.91)$$

$$\underset{\sim}{D}_{total} = \underset{\sim}{D} + \hat{\underset{\sim}{D}} \ , \qquad (2.92)$$

$$\underset{\sim}{H}_{total} = \underset{\sim}{H} + \hat{\underset{\sim}{H}} \ , \qquad (2.93)$$

where σ^α, $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{D}$ and $\underset{\sim}{H}$ arise from local (external applied) fields, while $\hat{\underset{\sim}{\sigma}}^\alpha$, $\hat{\underset{\sim}{E}}$, $\hat{\underset{\sim}{B}}$, $\hat{\underset{\sim}{D}}$ and $\hat{\underset{\sim}{H}}$ arise due to deformations induced by the external fields and the local fields; i.e.,

σ^{α} = local charge-currents due to response to external fields,

$\underset{\sim}{E}$ = local electric field due to response to external fields,

$\underset{\sim}{B}$ = local density of magnetic flux due to response to external fields,

$\underset{\sim}{D}$ = local electric displacement due to response to external fields,

$\underset{\sim}{H}$ = local magnetic field intensity due to response to external fields,

$\hat{\sigma}^{\alpha}$ = internally induced charge-current that arises because of the response to the local and external fields,

$\underset{\sim}{\hat{E}}$ = internally induced electric field that arises because of the response to the local and external fields,

$\underset{\sim}{\hat{B}}$ = internally induced density of magnetic flux that arises because of the response to the local and external fields,

$\underset{\sim}{\hat{D}}$ = internally induced electric displacement that arises because of the response to the local and external fields,

$\hat{\underset{\sim}{H}}$ = internally induced magnetic field intensity that arises because of the response to the local and external fields.

If we combine (2.33) and (2.35) and use (2.89), we see that

$$\sigma^{\alpha}_{(\text{total}),\alpha} = 0 \;. \tag{2.94}$$

Thus, in contrast to (2.33) and (2.35), we see that we obtain *conservation of total charge*.

We also define $\bar{\phi}^{\alpha\beta}_{(\text{total})}$ and $g^{\alpha\beta}_{(\text{total})}$ by

$$\bar{\phi}^{\alpha\beta}_{(\text{total})} = [\text{dual } \underset{\sim}{E}_{\text{total}}, \underset{\sim}{B}_{\text{total}}] \;, \tag{2.95}$$

$$g^{\alpha\beta}_{(\text{total})} = [\text{dual } \underset{\sim}{H}_{\text{total}}, -\underset{\sim}{D}_{\text{total}}] \;. \tag{2.96}$$

In view of (2.90), (2.91), (2.92) and (2.93), and definitions (2.46), (2.58), (2.71), (2.78), we see that (2.95) and (2.96) become

$$\bar{\phi}^{\alpha\beta}_{(\text{total})} = \bar{\phi}^{\alpha\beta} + \bar{\hat{\phi}}^{\alpha\beta} \;, \tag{2.97}$$

$$g^{\alpha\beta}_{(\text{total})} = g^{\alpha\beta} + \hat{g}^{\alpha\beta} \;. \tag{2.98}$$

If we use (2.95) to rewrite (2.62), we obtain

$$\bar{\phi}^{\alpha\beta}_{(\text{total}),\beta} = 0 \;, \tag{2.99}$$

and this is the general differential equation of *conservation of total magnetic flux*.

A substitution of (2.89) and (2.98) into (2.84) gives us

$$g^{\alpha\beta}_{(\text{total}),\beta} - \sigma^{\alpha}_{(\text{total})} = 0 \,, \qquad (2.100)$$

and this is the general differential equation for Ampere's and Gauss' Laws. Thus, Maxwell's equations take their classical form when expressed in terms of total field quantities.

In this same context, we note that the definitions

$$\mathcal{E}^{\alpha}_{(\text{total})} = \mathcal{E}^{\alpha} + \hat{\mathcal{E}}^{\alpha} \,, \qquad (2.101)$$

$$\mathcal{B}^{\alpha}_{(\text{total})} = \mathcal{B}^{\alpha} + \hat{\mathcal{B}}^{\alpha} \,, \qquad (2.102)$$

$$\mathcal{D}^{\alpha}_{(\text{total})} = \mathcal{D}^{\alpha} + \hat{\mathcal{D}}^{\alpha} \,, \qquad (2.103)$$

$$\mathcal{H}^{\alpha}_{(\text{total})} = \mathcal{H}^{\alpha} + \hat{\mathcal{H}}^{\alpha} \,, \qquad (2.104)$$

together with substituting (2.52), (2.61), (2.74) and (2.81) into (2.101), (2.102), (2.103) and (2.104) gives the following standard expressions of the classic theory for $\mathcal{E}^{\alpha}_{(\text{total})}$, $\mathcal{B}^{\alpha}_{(\text{total})}$, $\mathcal{D}^{\alpha}_{(\text{total})}$ and $\mathcal{H}^{\alpha}_{(\text{total})}$ in terms of $\underset{\sim}{E}_{\text{total}}$, $\underset{\sim}{B}_{\text{total}}$, $\underset{\sim}{H}_{\text{total}}$, $\underset{\sim}{D}_{\text{total}}$ and $\underset{\sim}{V}$:

$$\mathcal{E}^{\alpha}_{(\text{total})} = \left[\frac{\underset{\sim}{E}_{\text{total}} + \underset{\sim}{V}\times\underset{\sim}{B}_{\text{total}}}{\sqrt{1-v^2}} \,,\, \frac{\underset{\sim}{V}\cdot\underset{\sim}{E}_{\text{total}}}{\sqrt{1-v^2}}\right] , \qquad (2.105)$$

$$\mathcal{B}^{\alpha}_{(\text{total})} = \left[\frac{\underset{\sim}{B}_{\text{total}} - \underset{\sim}{V}\times\underset{\sim}{E}_{\text{total}}}{\sqrt{1-v^2}} \,,\, \frac{\underset{\sim}{V}\cdot\underset{\sim}{B}_{\text{total}}}{\sqrt{1-v^2}}\right] , \qquad (2.106)$$

$$\mathcal{D}^{\alpha}_{(\text{total})} = \left[\frac{\underset{\sim}{D}_{\text{total}} + \underset{\sim}{V} \times \underset{\sim}{H}_{\text{total}}}{\sqrt{1-v^2}}, \frac{\underset{\sim}{V} \cdot \underset{\sim}{D}_{\text{total}}}{\sqrt{1-v^2}}\right], \qquad (2.107)$$

$$\mathcal{H}^{\alpha}_{(\text{total})} = \left[\frac{\underset{\sim}{H}_{\text{total}} - \underset{\sim}{V} \times \underset{\sim}{D}_{\text{total}}}{\sqrt{1-v^2}}, \frac{\underset{\sim}{V} \cdot \underset{\sim}{H}_{\text{total}}}{\sqrt{1-v^2}}\right]. \qquad (2.108)$$

Thus, the nonlocal theory obtains from the classic theory by replacing all field variables by their total values; that is, by their local values + their nonlocal residual values. We take the view that this demonstrated decomposition of the total fields into a local part and a nonlocal residual part is fundamental to continuum electrodynamics.

3. The Heat Production Inequality

The second law of thermodynamics can be expressed by means of several different inequalities. The formulation of these inequalities is independent of the character of the media under consideration.

To formulate the relativistic extension of the Clausius-Duhem inequality, assign to every three-dimensional subspace $S^{(3)}$ of a material tube (β) a scalar invariant $H[S^{(3)}]$ of the form

$$H[S^{(3)}] = \int_{S^{(3)}} \eta^{\alpha} dS_{\alpha}^{(3)} .$$

The second law of thermodynamics can then be stated by the requirement that

$$\oint \eta^{\alpha} dS_{\alpha}^{(3)} + \int_{(\beta)} \gamma \, dV^{(4)} \geq 0 \tag{3.1}$$

for every material tube (β), where the scalar invariant γ is the supply of entropy from external sources.

It is convenient to decompose η^{α} into its space-like and time-like components

$$\eta^{\alpha} = \eta_{o} u^{\alpha} + S^{\alpha} , \tag{3.2}$$

where

$$\begin{aligned} \eta_{o} &\equiv -\eta^{\alpha} u_{\alpha} , \\ S^{\alpha} &\equiv S^{\alpha}_{\beta} \eta^{\beta} . \end{aligned} \tag{3.3}$$

We can then define a class of simple thermodynamic processes by

$$S^\alpha \equiv \frac{q^\alpha}{\theta} \;, \quad \gamma \equiv \frac{h_o}{\theta} \;, \quad h_o \equiv f^\mu_{(m)} u_\mu \;, \quad \theta > 0 \;, \qquad (3.4)$$

where

$q^\alpha \equiv$ heat flow four-vector ,

$h_o \equiv$ heat supply ,

$\theta \equiv$ temperature ,

$f^\mu_{(m)} \equiv$ external generalized mechanical four-vector body force .

The Green-Gauss theorem and (3.1) yield the global statement

$$\int_{(\beta)} (\eta^\alpha_{,\alpha} + \gamma)\, dV^{(4)} \geq 0 \;. \qquad (3.5)$$

We now define η_{oo} by

$$\eta_o = n_o \eta_{oo} \;. \qquad (3.6)$$

By use of equations (2.5), (3.2) and (3.6), we can replace inequality (3.5) by

$$\int_{(\beta)} (n_o D\eta_{oo} + S^\alpha_{,\alpha} + \gamma)\, dV^{(4)} \geq 0 \;. \qquad (3.7)$$

If the thermodynamic process is simple, as defined by (3.4), we can rewrite (3.7) as

$$\int_{(\beta)} \left[n_o D\eta_{oo} + \left(\frac{q^\alpha}{\theta}\right)_{,\alpha} + \frac{f^\alpha_{(m)} u_\alpha}{\theta} \right] dV^{(4)} \geq 0 \;. \qquad (3.8)$$

Inequality (3.8) states that the total internal production of entropy of any material tube in space-time is non-negative.

The nonlocal theory presented here is based upon the alternative inequality

$$\int_{(\beta)} \theta\left[n_o D\eta_{oo} + \left(\frac{q^\alpha}{\theta}\right)_{,\alpha} + \frac{f^\alpha_{(m)}u_\alpha}{\theta}\right] dV^{(4)} \geq 0 \ ; \quad (3.9)$$

that is, the total internal production of heat of the whole material tube (β) in space-time is non-negative. This statement can be localized by introducing the *heat supply residual* $\hat{S}_o$ (equivalence class) with the property

$$\int_{(\beta)} n_o \hat{S}_o dV^{(4)} \equiv 0 \ . \quad (3.10)$$

A combination of (3.9) and (3.10) then gives the following equivalent local inequality

$$n_o \theta D\eta_{oo} + \frac{q^\alpha}{\theta}\,\theta_{,\alpha} + q^\alpha_{,\alpha} + f^\alpha_{(m)}u_\alpha + n_o\hat{S}_o \geq 0 \ . \quad (3.11)$$

We now consider the electromagnetic interactions with ponderable matter. The local theories of electromagnetic interactions with ponderable matter were derived by Grot and Eringen [1]. They used the idea which originally came from Lorentz [10] to approach this problem by utilizing the electromagnetic field equations for ponderable matter.

We assume that the external body force f^μ can divide into two parts, one is mechanical force $f^{\mu}_{(m)}$, the other is electromagnetic force $f^{\mu}_{(e)}$, such that

$$f^{\mu} = f^{\mu}_{(m)} + f^{\mu}_{(e)} \; . \tag{3.12}$$

Here, we accept the arguments in Grot and Eringen [1], Maugin and Eringen [2], Dixon and Eringen [3] (neglecting electric quadruples and higher terms) so that we treat $f^{\mu}_{(e)}$ as

$$f^{\mu}_{(e)} = \pi^{\beta\alpha}\phi^{\mu}_{\alpha,\beta} + \sigma^{\alpha}\phi^{\mu}_{\alpha} \; . \tag{3.13}$$

We also assume the local body couple $L^{\alpha\beta}$ arises only due to the presence of the electromagnetic field so that

$$L^{\mu\nu} = L^{\mu\nu}_{(e)} = -\pi^{[\mu}_{\alpha}\phi^{\nu]\alpha} \; , \tag{3.14}$$

where $\pi^{\alpha\beta}$ is defined by

$$g^{\alpha\beta} \equiv \phi^{\alpha\beta} - \pi^{\alpha\beta} \; . \tag{3.15}$$

The *local polarization tensor* $\pi^{\alpha\beta}$ is a skew-symmetric tensor of the form

$$\pi^{\alpha\beta} = [\text{dual } \underset{\sim}{M} + \text{dual}(\underset{\sim}{V}\times\underset{\sim}{P}), \underset{\sim}{P}] \; , \tag{3.16}$$

where $\underset{\sim}{P}$ is the *local polarization vector* and $\underset{\sim}{M}$ is the *local magnetization vector*. This definition of $\pi^{\alpha\beta}$ is original with Lorentz [10].

The conservation of energy-momentum, equation (2.17), and the balance of moment of energy-momentum, equation (2.27), are now

$$T^{\mu\nu}{}_{,\nu} - (f^{\mu}_{(m)}+f^{\mu}_{(e)}) = n_o\hat{f}^{\mu} , \qquad (3.17)$$

$$S^{\alpha\beta\mu}{}_{,\mu} - T^{[\alpha\beta]} - L^{\alpha\beta}_{(e)} - n_o x^{[\alpha}\hat{f}^{\beta]} = \hat{\ell}^{\alpha\beta} . \qquad (3.18)$$

It is thus natural to define $\hat{f}^{\mu}_{(m)}$ and $\hat{f}^{\mu}_{(e)}$ by

$$n_o\hat{f}^{\mu} = n_o\hat{f}^{\mu}_{(m)} + \hat{f}^{\mu}_{(e)} , \qquad (3.19)$$

where, $n_o\hat{f}^{\mu}_{(m)}$ is the *nonlocal mechanical body force and energy supply* per unit volume and $\hat{f}^{\mu}_{(e)}$ is the *nonlocal electromagnetic body force.*

Further, we can decompose $n_o\hat{f}^{\mu}_{(m)}$ into

$$n_o\hat{f}^{\mu}_{(m)} = n_o\hat{f}_{(mo)}u^{\mu} + n_o\hat{f}^{\mu}_{(m3)}$$

where

$$\begin{aligned} n_o\hat{f}_{(mo)} &\equiv -n_o\hat{f}^{\mu}_{(m)}u_{\mu} , \\ n_o\hat{f}^{\mu}_{(m3)} &\equiv S^{\mu}_{\beta}n_o\hat{f}^{\beta}_{(m)} , \quad n_o\hat{f}^{\mu}_{(m3)}u_{\mu} = 0 . \end{aligned} \qquad (3.20)$$

This definition is similar to the definition in (2.16). We also note that since $n_o\hat{f}_{(mo)}$ and $n_o\hat{f}^{\mu}_{(m3)}$ are the time-like and space-like components of $n_o\hat{f}^{\mu}_{(m)}$ represently, both $n_o\hat{f}_{(mo)}$ and $n_o\hat{f}^{\mu}_{(m3)}$ satisfy the zero mean conditions

$$\begin{aligned} \int_{(\beta)} n_o\hat{f}_{(mo)}dV^{(4)} &\equiv 0 , \\ \int_{(\beta)} n_o\hat{f}^{\mu}_{(m3)}dV^{(4)} &\equiv 0 . \end{aligned} \qquad (3.21)$$

Now, *we assume that the nonlocal electromagnetic body force* $\hat{f}^{\mu}_{(e)}$ *exists due to the presence of the nonlocal electromagnetic fields in the form similar to the local electromagnetic force* (equation (3.13)),

$$\hat{f}^{\mu}_{(e)} = \hat{\pi}^{\beta\alpha}\hat{\phi}^{\mu}_{\alpha,\beta} + \hat{\sigma}\hat{\phi}^{\mu}_{\alpha} \, , \tag{3.22}$$

where $\hat{\sigma}^{\alpha}$ and $\hat{\phi}_{\alpha\beta}$ are defined by (2.36) and (2.59) represently, and $\hat{\pi}^{\alpha\beta}$ is defined by

$$\hat{g}^{\alpha\beta} \equiv \hat{\phi}^{\alpha\beta} - \hat{\pi}^{\alpha\beta} \, . \tag{3.23}$$

The *nonlocal polarization tensor* $\hat{\pi}^{\alpha\beta}$ is a skew-symmetric tensor of the form

$$\hat{\pi}^{\alpha\beta} = [\text{dual } \hat{\underset{\sim}{M}} + \text{dual}(\hat{\underset{\sim}{V}}\times\hat{\underset{\sim}{P}}),\hat{\underset{\sim}{P}}] \, , \tag{3.24}$$

where $\hat{\underset{\sim}{P}}$ is the *nonlocal polarization vector* and $\hat{\underset{\sim}{M}}$ is the *nonlocal magnetization vector.*

In the local theories of electromagnetic interactions with matter, we can assume that $S^{\alpha\beta\mu}=0$ and $L^{\alpha\beta}_{(m)}=0$ (cf. Grot and Eringen [1]). In a relativistic generalization of mechanical theories of couple stresses, it is necessary to include the spin tensor $S^{\alpha\beta\mu}$ both in the local case (cf. Maugin and Eringen [2]) and in the nonlocal case (a fuller investigation of the properties of this tensor in nonlocal theories is left for further research). Here, we simply assume that *the local spin tensor gradient* $S^{\alpha\beta\mu}{}_{,\mu}$ *exists only* *due to the effects of the nonlocal mechanical body torque* $n_o x^{[\alpha}\hat{f}^{\beta]}_{(m)}$ *and the nonlocal electromagnetic body torque* $x^{[\alpha}\hat{f}^{\beta]}_{(e)}$ *such that*

$$S^{\alpha\beta\mu}{}_{,\mu} = n_o x^{[\alpha}\hat{f}^{\beta]}_{(m)} + x^{[\alpha}\hat{f}^{\beta]}_{(e)}$$
$$= n_o x^{[\alpha}\hat{f}^{\beta]} \quad . \tag{3.25}$$

This expression indicates that the local spin tensor gradient $S^{\alpha\beta\mu}{}_{,\mu}$ exists only due to the generation of nonlocal body torque.

In addition, we also assume that *the nonlocal body couple* $\hat{\ell}^{\alpha\beta}$ *exists only due to the presence of the nonlocal electromagnetic field so that*

$$\hat{\ell}^{\alpha\beta} = \hat{L}^{\alpha\beta}_{(m)} + \hat{L}^{\alpha\beta}_{(e)} \ , \tag{3.26}$$

where $\hat{L}^{\alpha\beta}_{(m)}=0$ and $\hat{L}^{\alpha\beta}_{(e)}$ is defined by

$$\hat{L}^{\alpha\beta}_{(e)} = -\hat{\pi}^{[\alpha}_{\mu}\hat{\phi}^{\beta]\mu} \quad . \tag{3.27}$$

Now, substituting (3.13), (3.14), (3.19), (3.22), (3.25) and (3.27) into (3.17) and (3.18), we obtain

$$T^{\mu\nu}{}_{,\nu} = (\pi^{\beta\alpha}\phi^{\mu}_{\alpha,\beta}+\hat{\pi}^{\beta\alpha}\hat{\phi}^{\mu}_{\alpha,\beta}) + (\sigma^{\alpha}\phi^{\mu}_{\alpha}+\hat{\sigma}^{\alpha}\hat{\phi}^{\mu}_{\alpha})$$
$$+ (f^{\mu}_{(m)}+n_o\hat{f}^{\mu}_{(m)}) \ , \tag{3.28}$$

$$T^{[\alpha\beta]} = \pi^{[\alpha}_{\mu}\phi^{\beta]\mu} + \hat{\pi}^{[\alpha}_{\mu}\hat{\phi}^{\beta]\mu} \quad . \tag{3.29}$$

In view of (2.50) and (2.59), it is useful to decompose $\pi^{\alpha\beta}$ and $\hat{\pi}^{\alpha\beta}$ into

$$\pi^{\alpha\beta} = \mathcal{P}^{\alpha}u^{\beta} - \mathcal{P}^{\beta}u^{\alpha} + \varepsilon^{\alpha\beta\gamma\delta}\mathcal{M}_{\gamma}u_{\delta} , \tag{3.30}$$

$$\hat{\pi}^{\alpha\beta} = \hat{\mathcal{P}}^{\alpha}u^{\beta} - \hat{\mathcal{P}}^{\beta}u^{\alpha} + \varepsilon^{\alpha\beta\gamma\delta}\hat{\mathcal{M}}_{\gamma}u_{\delta} , \tag{3.31}$$

where

$$\mathcal{P}^{\alpha} \equiv \pi^{\alpha\beta}u_{\beta} , \quad \mathcal{P}^{\beta}u_{\beta} = 0 ,$$

$$\mathcal{M}_{\alpha} \equiv \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\pi^{\beta\gamma}u^{\delta} , \quad \mathcal{M}^{\beta}u_{\beta} = 0 , \tag{3.32}$$

$$\hat{\mathcal{P}}^{\alpha} \equiv \hat{\pi}^{\alpha\beta}u_{\beta} , \quad \hat{\mathcal{P}}^{\beta}u_{\beta} = 0 ,$$

$$\hat{\mathcal{M}}_{\alpha} \equiv \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\hat{\pi}^{\beta\gamma}u^{\delta} \quad \hat{\mathcal{M}}^{\beta}u_{\beta} = 0 . \tag{3.33}$$

Using (2.50) and (2.72) we see that (3.15) is equivalent to

$$\mathcal{D}^{\alpha} = \mathcal{E}^{\alpha} + \mathcal{P}^{\alpha} , \quad \mathcal{H}^{\alpha} = \mathcal{B}^{\alpha} - \mathcal{M}^{\alpha} . \tag{3.34}$$

Thus (2.59) and (2.79) show that (3.23) is equivalent to

$$\hat{\mathcal{D}}^{\alpha} = \hat{\mathcal{E}}^{\alpha} + \hat{\mathcal{P}}^{\alpha} , \quad \hat{\mathcal{H}}^{\alpha} = \hat{\mathcal{B}}^{\alpha} - \hat{\mathcal{M}}^{\alpha} . \tag{3.35}$$

From (2.50) and (3.30), (2.59) and (3.31) we can see that (cf. Grot and Eringen [1])

$$\begin{aligned}\pi^{\alpha\beta}\phi_{\mu\beta} = &- \mathcal{P}^{\beta}\mathcal{E}_{\beta}u^{\alpha}u_{\mu} + \overset{*}{\pi}{}^{\alpha\beta}\mathcal{E}_{\beta}u_{\mu} + \mathcal{P}^{\alpha}u_{\mu} + \mathcal{P}^{\alpha}\mathcal{E}_{\mu} \\ &- \overset{*}{\phi}_{\alpha\beta}\mathcal{P}^{\beta}u^{\alpha} + S^{\alpha}_{\mu}\mathcal{B}^{\beta}\mathcal{M}_{\beta} - \mathcal{B}^{\alpha}\mathcal{M}_{\mu} ,\end{aligned} \tag{3.36}$$

$$\begin{aligned}\hat{\pi}^{\alpha\beta}\hat{\phi}_{\mu\beta} = &- \hat{\mathcal{P}}^{\beta}\hat{\mathcal{E}}_{\beta}u^{\alpha}u_{\mu} + \hat{\overset{*}{\pi}}{}^{\alpha\beta}\hat{\mathcal{E}}_{\beta}u_{\mu} + \hat{\mathcal{P}}^{\alpha}u_{\mu} + \hat{\mathcal{P}}^{\alpha}\hat{\mathcal{E}}_{\mu} \\ &- \hat{\overset{*}{\phi}}_{\mu\beta}\hat{\mathcal{P}}^{\beta}u^{\alpha} + S^{\alpha}_{\mu}\hat{\mathcal{B}}^{\beta}\hat{\mathcal{M}}_{\beta} - \hat{\mathcal{B}}^{\alpha}\hat{\mathcal{M}}_{\mu} ,\end{aligned} \tag{3.37}$$

where

$$
\begin{aligned}
\overset{*}{\pi}^{\alpha\beta} &\equiv \varepsilon^{\alpha\beta\gamma\delta}\mathcal{M}_\gamma u_\delta \equiv S^\alpha_\gamma S^\beta_\delta \pi^{\gamma\delta} , \quad \overset{*}{\pi}^{\alpha\beta}u_\beta = 0 , \\
\hat{\overset{*}{\pi}}{}^{\alpha\beta} &\equiv \varepsilon^{\alpha\beta\gamma\delta}\hat{\mathcal{M}}_\gamma u_\delta \equiv S^\alpha_\gamma S^\beta_\delta \hat{\pi}^{\gamma\delta} , \quad \hat{\overset{*}{\pi}}{}^{\alpha\beta}u_\beta = 0 , \\
\overset{*}{\phi}_{\alpha\beta} &\equiv \varepsilon^{\alpha}_{\alpha\beta\gamma\delta}\mathcal{B}^\gamma u^\delta \equiv S^\gamma_\alpha S^\delta_\beta \phi_{\gamma\delta} , \quad \overset{*}{\phi}_{\alpha\beta}u^\beta = 0 , \\
\hat{\overset{*}{\phi}}_{\alpha\beta} &\equiv \varepsilon_{\alpha\beta\gamma\delta}\hat{\mathcal{B}}^\gamma u^\delta \equiv S^\gamma_\alpha S^\delta_\beta \hat{\phi}_{\gamma\delta} , \quad \hat{\overset{*}{\phi}}_{\alpha\beta}u^\beta = 0 .
\end{aligned}
\tag{3.38}
$$

If we take the skew-symmetric parts of (3.36) and (3.37), we obtain

$$
\pi_\beta^{[\alpha}\phi^{\mu]\beta} = -\wp^{[\alpha}\mathcal{E}^{\mu]} + \mathcal{E}_\beta\overset{*}{\pi}^{\beta[\alpha}u^{\mu]} - \mathcal{P}_\beta\overset{*}{\phi}^{\beta[\mu}u^{\alpha]} - \mathcal{M}^{[\alpha}\mathcal{B}^{\mu]} , \tag{3.39}
$$

$$
\hat{\pi}_\beta^{[\alpha}\hat{\phi}^{\mu]\beta} = -\hat{\wp}^{[\alpha}\hat{\mathcal{E}}^{\mu]} + \hat{\mathcal{E}}_\beta\hat{\overset{*}{\pi}}{}^{\beta[\alpha}u^{\mu]} - \hat{\mathcal{P}}_\beta\hat{\overset{*}{\phi}}{}^{\beta[\mu}u^{\alpha]} - \hat{\mathcal{M}}^{[\alpha}\hat{\mathcal{B}}^{\mu]} . \tag{3.40}
$$

Similarly, the skew-symmetric part of (2.11), gives us

$$
T^{[\alpha\beta]} = u^{[\alpha}q^{\beta]} + p^{[\alpha}u^{\beta]} - t^{[\alpha\beta]} . \tag{3.41}
$$

Substituting (3.39), (3.40) and (3.41) into (3.29), we finally obtain

$$
\begin{aligned}
p^{[\alpha}u^{\beta]} - q^{[\alpha}u^{\beta]} - t^{[\alpha\beta]} &= (\mathcal{P}_\gamma\overset{*}{\phi}^{\gamma[\alpha}u^{\beta]} + \hat{\mathcal{P}}_\gamma\hat{\overset{*}{\phi}}{}^{\gamma[\alpha}u^{\beta]}) \\
&+ (\mathcal{E}_\gamma\overset{*}{\pi}^{\gamma[\alpha}u^{\beta]} + \hat{\mathcal{E}}_\gamma\hat{\overset{*}{\pi}}{}^{\gamma[\alpha}u^{\beta]}) - (\wp^{[\alpha}\mathcal{E}^{\beta]} + \hat{\wp}^{[\alpha}\hat{\mathcal{E}}^{\beta]}) \\
&- (\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]} + \hat{\mathcal{M}}^{[\alpha}\hat{\mathcal{B}}^{\beta]}) ,
\end{aligned}
\tag{3.42}
$$

since

$$
u^{[\alpha}q^{\beta]} = -q^{[\alpha}u^{\beta]} , \quad \mathcal{P}_\gamma\overset{*}{\phi}^{\gamma[\beta}u^{\alpha]} = -\mathcal{P}_\gamma\overset{*}{\phi}^{\gamma[\alpha}u^{\beta]}
$$

and

$$\hat{\mathcal{P}}_\gamma \hat{\overset{*}{\phi}}{}^{\gamma[\beta}u^{\alpha]} = -\hat{\mathcal{P}}_\gamma \hat{\overset{*}{\phi}}{}^{\gamma[\alpha}u^{\beta]} .$$

By using the properties (2.13), (2.25), (2.48), (2.49), (3.20), (3.32), (3.33) and (3.38) and multiplication of (3.42) by u_α and S^γ_α, we see that

$$p^\alpha = q^\alpha + (\mathcal{P}_\gamma \overset{*}{\phi}{}^{\gamma\alpha} + \hat{\mathcal{P}}_\gamma \hat{\overset{*}{\phi}}{}^{\gamma\alpha}) + (\mathcal{E}_\gamma \overset{*}{\pi}{}^{\gamma\alpha} + \hat{\mathcal{E}}_\gamma \overset{*}{\pi}{}^{\gamma\alpha}) , \quad (3.43)$$

$$t^{[\alpha\beta]} = (\mathcal{P}^{[\alpha}\mathcal{E}^{\beta]} + \hat{\mathcal{P}}^{[\alpha}\hat{\mathcal{E}}^{\beta]}) + (\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]} + \hat{\mathcal{M}}^{[\alpha}\hat{\mathcal{B}}^{\beta]}) . \quad (3.44)$$

Using (2.5), (2.30), (2.36), (2.48), (2.49), (2.50), (3.30), (3.31), (3.32), (3.33) and (3.39), we can show that

$$\sigma^\alpha \phi^\mu_\alpha u_\mu = -j^\alpha \mathcal{E}_\alpha , \quad (3.45)$$

$$\hat{\sigma}^\alpha \hat{\phi}^\mu_\alpha u_\mu = -\hat{j}^\alpha \hat{\mathcal{E}}_\alpha , \quad (3.46)$$

$$u_\mu \pi^{\beta\alpha} \phi^\mu_{\alpha,\beta} = \mathcal{P}^\alpha D\mathcal{E}_\alpha + \mathcal{P}^\alpha \overset{*}{\phi}_{\beta\alpha} Du^\beta + \mathcal{M}^\alpha D\mathcal{B}_\alpha + \mathcal{E}_\alpha \overset{*}{\pi}{}^{\beta\alpha} Du_\beta , \quad (3.47)$$

$$u_\mu \hat{\pi}^{\beta\alpha} \phi^\mu_{\alpha,\beta} = \hat{\mathcal{P}}^\alpha D\hat{\mathcal{E}}_\alpha + \hat{\mathcal{P}}^\alpha \hat{\overset{*}{\phi}}_{\beta\alpha} Du^\beta + \hat{\mathcal{M}}^\alpha D\hat{\mathcal{B}}_\alpha + \hat{\mathcal{E}}_\alpha \overset{*}{\pi}{}^{\beta\alpha} Du_\beta . \quad (3.48)$$

We now substitute (3.12), (3.13), (3.19), (3.20) and (3.22) into (2.19) to obtain

$$n_o D\varepsilon + q^\beta_{;\beta} + p^\beta Du_\beta - t^{\alpha\beta}\overset{*}{u}_{\alpha\beta} = -(\pi^{\beta\alpha}\phi^\mu_{\alpha,\beta} + \hat{\pi}^{\beta\alpha}\hat{\phi}^\mu_{\alpha,\beta})u_\mu - (\sigma^\alpha\phi^\mu_\alpha + \hat{\sigma}^\alpha\hat{\phi}^\mu_\alpha)u_\mu - (f^\alpha_{(m)}u_\alpha - n_o\hat{f}_{(mo)}) . \quad (3.49)$$

Here, we introduce

$$\overset{*}{u}{}^{\alpha}_{\beta} \equiv S^{\mu}_{\beta} u^{\alpha}_{,\mu} \ , \tag{3.50}$$

from which it is obvious that $\overset{*}{u}_{\alpha\beta} = u_{\alpha,\beta}$.

Substituting (3.45), (3.46), (3.47) and (3.48) into the right-hand side of (3.49), we obtain

$$\begin{aligned} n_o D\varepsilon + q^{\beta}_{,\beta} + p^{\beta} Du_{\beta} - t^{\alpha\beta}\overset{*}{u}_{\alpha\beta} &= -\mathcal{P}^{\alpha} D\mathcal{E}_{\alpha} - \mathcal{P}^{\alpha}\overset{*}{\phi}_{\beta\alpha} Du^{\beta} \\ &- \mathcal{M}^{\alpha} D\mathcal{B}_{\alpha} - \mathcal{E}_{\alpha}\overset{*}{\pi}{}^{\beta\alpha} Du_{\beta} + j^{\alpha}\mathcal{E}_{\alpha} - f^{\alpha}_{(m)} u_{\alpha} + n_o \hat{f}_{(mo)} \\ &+ (-\hat{\mathcal{P}}^{\alpha} D\hat{\mathcal{E}}_{\alpha} - \hat{\mathcal{P}}^{\alpha}\hat{\overset{*}{\phi}}_{\beta\alpha} Du^{\beta} - \hat{\mathcal{M}}^{\alpha} D\hat{\mathcal{B}}_{\alpha} - \hat{\mathcal{E}}_{\alpha}\hat{\overset{*}{\pi}}{}^{\beta\alpha} Du_{\beta} + \hat{j}^{\alpha}\hat{\mathcal{E}}_{\alpha}) \ . \end{aligned} \tag{3.51}$$

Now, we introduce the relativistic generalization of the deformation rate tensor $\overset{*}{d}_{\alpha\beta}$ by

$$\overset{*}{d}_{\alpha\beta} \equiv \frac{1}{2}\,(\overset{*}{u}_{\alpha\beta} + \overset{*}{u}_{\beta\alpha}) \equiv \overset{*}{u}_{(\alpha\beta)} \ , \tag{3.52}$$

and the relativistic generalization of spin $\overset{*}{\omega}_{\alpha\beta}$ by

$$\overset{*}{\omega}_{\alpha\beta} \equiv \frac{1}{2}\,(\overset{*}{u}_{\alpha\beta} + \overset{*}{u}_{\beta\alpha}) \equiv \overset{*}{u}_{[\alpha\beta]} \ , \tag{3.53}$$

so that

$$\overset{*}{u}_{\alpha\beta} = \overset{*}{d}_{\alpha\beta} + \overset{*}{\omega}_{\alpha\beta} \ . \tag{3.54}$$

Using the definition (3.52), (3.53) and (3.54), we obtain

$$t^{\alpha\beta}\overset{*}{u}_{\alpha\beta} = t^{(\alpha\beta)}\overset{*}{d}_{\alpha\beta} + t^{[\alpha\beta]}\overset{*}{\omega}_{\alpha\beta} \ . \tag{3.55}$$

From (3.44), we see that equation (3.55) becomes

$$\begin{aligned} t^{\alpha\beta}\overset{*}{u}_{\alpha\beta} = t^{(\alpha\beta)}\overset{*}{d}_{\alpha\beta} + [(\mathcal{P}^{[\alpha}\mathcal{E}^{\beta]} + \hat{\mathcal{P}}^{[\alpha}\hat{\mathcal{E}}^{\beta]}) + (\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]} \\ + \hat{\mathcal{M}}^{[\alpha}\hat{\mathcal{B}}^{\beta]})]\overset{*}{\omega}_{\alpha\beta} \ . \end{aligned} \tag{3.56}$$

We also note that

$$(\mathcal{P}^{\alpha}\mathcal{E}^{\beta}+\hat{\mathcal{P}}^{\alpha}\hat{\mathcal{E}}^{\beta}+\mathcal{M}^{\alpha}\mathcal{B}^{\beta}+\hat{\mathcal{M}}^{\alpha}\hat{\mathcal{B}}^{\beta})u_{\alpha,\beta}$$

$$= (\mathcal{P}^{[\alpha}\mathcal{E}^{\beta]}+\hat{\mathcal{P}}^{[\alpha}\hat{\mathcal{E}}^{\beta]}+\mathcal{M}^{[\alpha}\mathcal{B}^{\beta]}+\hat{\mathcal{M}}^{[\alpha}\hat{\mathcal{B}}^{\beta]})\overset{*}{\omega}_{\alpha\beta}$$

$$+ (\mathcal{P}^{(\alpha}\mathcal{E}^{\beta)}+\hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)}+\mathcal{M}^{(\alpha}\mathcal{B}^{\beta)}+\hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)})\overset{*}{d}_{\alpha\beta} \; . \qquad (3.57)$$

Substituting (3.57) into (3.56), shows that we have

$$t^{\alpha\beta}\overset{*}{u}_{\alpha\beta} = [t^{(\alpha\beta)}-\mathcal{P}^{(\alpha}\mathcal{E}^{\beta)}-\hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)}-\mathcal{M}^{(\alpha}\mathcal{B}^{\beta)}-\hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)}]\overset{*}{d}_{\alpha\beta}$$

$$+ (\mathcal{P}^{\alpha}\mathcal{E}^{\beta}+\hat{\mathcal{P}}^{\alpha}\hat{\mathcal{E}}^{\beta}+\mathcal{M}^{\alpha}\mathcal{B}^{\beta}+\hat{\mathcal{M}}^{\alpha}\hat{\mathcal{B}}^{\beta})u_{\alpha,\beta} \; , \qquad (3.58)$$

while substituting (3.43) and (3.58) into (3.51), yields

$$n_o D\varepsilon + q^{\beta}_{,\beta} + q^{\beta}Du_{\beta}$$

$$- [t^{(\alpha\beta)}-\mathcal{P}^{(\alpha}\mathcal{E}^{\beta)}-\mathcal{M}^{(\alpha}\mathcal{B}^{\beta)}-\hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)}-\hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)}]\overset{*}{d}_{\alpha\beta}$$

$$+ \mathcal{P}^{\alpha}[D\mathcal{E}_{\alpha}-\mathcal{E}^{\beta}u_{\alpha,\beta}] + \mathcal{M}^{\alpha}[D\mathcal{B}_{\alpha}-\mathcal{B}^{\beta}u_{\alpha,\beta}]$$

$$+ \hat{\mathcal{P}}^{\alpha}[D\hat{\mathcal{E}}_{\alpha}-\hat{\mathcal{E}}^{\beta}u_{\alpha,\beta}] + \hat{\mathcal{M}}^{\alpha}[D\hat{\mathcal{B}}_{\alpha}-\hat{\mathcal{B}}^{\beta}u_{\alpha,\beta}]$$

$$- (j^{\alpha}\mathcal{E}_{\alpha}+\hat{j}^{\alpha}\hat{\mathcal{E}}_{\alpha}) - n_o\hat{f}_{(mo)} = - f^{\alpha}_{(m)}u_{\alpha} \; . \qquad (3.59)$$

We now introduce independent variable $\overset{*}{\theta}_{\alpha}$ by

$$\overset{*}{\theta}_{\alpha} \equiv S^{\beta}_{\alpha}(\theta_{,\beta}+\theta Du_{\beta}) \; , \qquad (3.60)$$

so that

$$\overset{*}{\theta}_{\alpha} = \theta_{,\alpha} + \theta Du_{\alpha} \; .$$

Substituting (3.59) into (3.11) and employing the new variable $\overset{*}{\theta}_\alpha$, we obtain the local form of the heat production inequality

$$n_o(\theta D\eta_{oo}-D\varepsilon) - \left(\frac{q^\alpha}{\theta}\right)\overset{*}{\theta}_\alpha + [t^{(\alpha\beta)} - \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} - \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} - \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)}$$

$$- \hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)}]\overset{*}{d}_{\alpha\beta} - \mathcal{P}^\alpha[D\mathcal{E}_\alpha - \mathcal{E}^\beta u_{\alpha,\beta}] - \mathcal{M}^\alpha[D\mathcal{B}_\alpha - \mathcal{B}^\beta u_{\alpha,\beta}]$$

$$- \hat{\mathcal{P}}^\alpha[D\hat{\mathcal{E}}_\alpha - \hat{\mathcal{E}}^\beta u_{\alpha,\beta}] - \hat{\mathcal{M}}^\alpha[D\hat{\mathcal{B}}_\alpha - \hat{\mathcal{B}}^\beta u_{\alpha,\beta}] + j^\alpha\mathcal{E}_\alpha + \hat{j}^\alpha\hat{\mathcal{E}}_\alpha$$

$$+ n_o(\hat{f}_{(mo)} + \hat{S}_o) \geq 0 \ . \qquad (3.61)$$

This inequality is fundamental in obtaining nonlocal constitutive equations for ponderable matter subject to electromagnetic interactions.

Similar to the end of section 2, we can also draw a very important conclusion concerning the presence of local and nonlocal polarization and magnetization fields. We define $\underset{\sim}{P}_{total}$ and $\underset{\sim}{M}_{total}$ by

$$\underset{\sim}{P}_{total} = \underset{\sim}{P} + \hat{\underset{\sim}{P}} \ , \qquad (3.62)$$

$$\underset{\sim}{M}_{total} = \underset{\sim}{M} + \hat{\underset{\sim}{M}} \ , \qquad (3.63)$$

where $\underset{\sim}{P}$ and $\underset{\sim}{M}$ arise from local (external effects) fields, while $\hat{\underset{\sim}{P}}$ and $\hat{\underset{\sim}{M}}$ arise due to deformations induced by the external fields and the local fields; i.e.

$\underset{\sim}{P}$ = local polarization vector due to response to external fields,

$\underset{\sim}{M}$ = local magnetization vector due to response to external fields,

$\hat{\underset{\sim}{P}}$ = internally induced polarization vector that arises because of the response to the local and external fields,

$\hat{\underset{\sim}{M}}$ = internally induced magnetization vector that arises because of the response to the local and external fields.

We also define $\pi^{\alpha\beta}_{(\text{total})}$ by

$$\pi^{\alpha\beta}_{(\text{total})} = [\text{dual } \underset{\sim}{M}_{\text{total}} + \text{dual}(\underset{\sim}{V} \times \underset{\sim}{P}_{\text{total}}), \underset{\sim}{P}_{\text{total}}] \ . \tag{3.64}$$

In view of (3.62), (3.63), and definitions (3.16), (3.24), we see that (3.64) becomes

$$\pi^{\alpha\beta}_{(\text{total})} = \pi^{\alpha\beta} + \hat{\pi}^{\alpha\beta} \ . \tag{3.65}$$

If we define $\phi^{\alpha\beta}_{(\text{total})}$ by

$$\phi^{\alpha\beta}_{(\text{total})} \equiv \text{dual } \bar{\phi}^{\alpha\beta}_{(\text{total})} \ , \tag{3.66}$$

we see that

$$\phi^{\alpha\beta}_{(\text{total})} = [\text{dual } \underset{\sim}{B}_{\text{total}}, \underset{\sim}{E}_{\text{total}}] \ . \tag{3.67}$$

In view of (2.44), (2.56), (2.90), (2.91), we see that (3.67) becomes

$$\phi^{\alpha\beta}_{(total)} = \phi^{\alpha\beta} + \hat{\phi}^{\alpha\beta} . \tag{3.68}$$

By use of the definitions (3.15), (3.23), and employing the results of (2.98), (3.65) and (3.68), we obtain

$$g^{\alpha\beta}_{(total)} = \phi^{\alpha\beta}_{(total)} - \pi^{\alpha\beta}_{(total)} . \tag{3.69}$$

In this same content, we note that the definition

$$\mathcal{P}^{\alpha}_{(total)} = \mathcal{P}^{\alpha} + \hat{\mathcal{P}}^{\alpha} , \tag{3.70}$$

$$\mathcal{M}^{\alpha}_{(total)} = \mathcal{M}^{\alpha} + \hat{\mathcal{M}}^{\alpha} , \tag{3.71}$$

together with substituting (2.101), (2.102), (2.103), (2.104), (3.70) and (3.71) into (3.34) and (3.35), we obtain

$$\mathcal{D}^{\alpha}_{(total)} = \mathcal{E}^{\alpha}_{(total)} + \mathcal{P}^{\alpha}_{(total)} , \tag{3.72}$$

$$\mathcal{H}^{\alpha}_{(total)} = \mathcal{B}^{\alpha}_{(total)} + \mathcal{M}^{\alpha}_{(total)} . \tag{3.73}$$

Thus, our previously noted view of the fundamental character of the decomposition of the total fields into a local part and a nonlocal residual part can be extended to the properties of material bodies that are characterized by polarization and magnetization.

4. Reformulation in Terms of the Helmholtz Free Energy and the Resulting Functional Inequality

Since the heat production inequality (3.61) shows that $\overset{*}{\theta}_\beta$ are required as independent variables, it is a definite convenience if we use temperature, θ, rather than entropy, η_{oo}, as an independent state parameter. To this purpose, introduce the Helmholtz free energy Ψ by (cf. Edelen and Laws [11])

$$\Psi = \varepsilon - \theta\eta_{oo} . \tag{4.1}$$

Let us also define the operator $\bar{D}$ by

$$\begin{aligned} \bar{D}Q^\alpha &\equiv DQ^\alpha - Q^\beta u^\alpha_{;\beta} , \\ \bar{D}Q_\alpha &= DQ_\alpha - Q^\beta u_{\alpha,\beta} . \end{aligned} \tag{4.2}$$

When we substitute (4.1) into (3.61) and use (4.2), we obtain the local inequality

$$\begin{aligned} &-n_o(D\Psi+\eta_{oo}D\theta) - \frac{q^\alpha}{\theta}\overset{*}{\theta}_\alpha + [t^{(\alpha\beta)} - \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} - \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} - \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)} \\ &-\hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)}]\overset{*}{d}_{\alpha\beta} - \mathcal{P}^\alpha\bar{D}\mathcal{E}_\alpha - \hat{\mathcal{P}}^\alpha\bar{D}\hat{\mathcal{E}}_\alpha - \mathcal{M}^\alpha\bar{D}\mathcal{B}_\alpha - \hat{\mathcal{M}}^\alpha\bar{D}\hat{\mathcal{B}}_\alpha \\ &+ j^\alpha\mathcal{E}_\alpha + \hat{j}^\alpha\hat{\mathcal{E}}_\alpha + n_o(\hat{f}_{(mo)}+\hat{S}_o) \geq 0 . \end{aligned} \tag{4.3}$$

We now assume that Π_Λ, $\Lambda=1,\ldots,N$ designate an additional N-tuple of arguments of Ψ that may occur as consequence of the dynamic process of the material tube (β) under examination. We also introduce the quantities

$$\mathcal{E}^K = X^K_{,\alpha}\mathcal{E}^\alpha \ , \qquad \hat{\mathcal{E}}^K = X^K_{,\alpha}\hat{\mathcal{E}}^\alpha \ ,$$

$$\mathcal{B}^K = X^K_{,\alpha}\mathcal{B}^\alpha \mathrm{sgn}(x^i/X^K) \ , \qquad \hat{\mathcal{B}}^K = X^K_{,\alpha}\hat{\mathcal{B}}^\alpha \mathrm{sgn}(x^i/X^K) \ ,$$

$$\theta^K = X^K_{,\alpha}\overset{*}{\theta}{}^\alpha \ . \tag{4.4}$$

Here x^i signify the spatial coordinates of the space-time event and X^K signify the coordinates of the reference state; the relation between them being

$$X^K = X^K(x^\alpha) \ ,$$

$$x^i = x^i(X^K, x^4) \ . \tag{4.5}$$

Consider a tensor function $\xi = \xi(x^\mu)$, we defined

$$\xi_{,\beta} \equiv \frac{\partial \xi(x^\alpha)}{\partial x^\beta} \ , \tag{4.6}$$

so that

$$X^K_{,\beta} \equiv \frac{\partial X^K(x^\alpha)}{\partial x^\beta} \ . \tag{4.7}$$

We explicitly assume that $\det(x^k_{,K}) \neq 0$, where

$$x^k_{,K} \equiv \frac{\partial x^k(X^K, x^4)}{\partial X^K} \ . \tag{4.8}$$

We also define

$$x^\alpha_K \equiv S^\alpha_k x^k_{,K} \ , \tag{4.9}$$

so that the quantities x^α_K (K fixed) are four vectors. Here $\mathrm{sgn}(x^i/X^K) \equiv \mathrm{sgn}[\det(x^k_{,K})]$, whose occurrence in (4.4) shows that $\mathcal{B}^\beta$ and $\hat{\mathcal{B}}^\beta$ remain axial vectors.

In order to proceed further, we note that nonlocal theories involve functional rather than function dependences in the constitutive relations. Accordingly, the natural setting becomes a function space rather than a finite dimensional vector space of values of a finte number of functions at a point in their mutual domain of definition. (cf. Edelen and Laws [11], Edelen, Green and Laws [12], Green and Laws [13], Eringen and Edelen [9], Eringen [14], Edelen [4].)

For convenience, we choose a Hilbert space as the underlying function space throughout the discussion (cf. Edelen [4]). Let $L_2(\beta)$ denote the Hilbert space of square summable functions on (β) with

$$\begin{aligned} ||u(x^\alpha)||^2 &= \int_{(\beta)} u^2(x^\alpha)dV^{(4)}(x^\alpha) , \\ \left\langle u(x^\alpha), v(x^\alpha)\right\rangle &= \int_{(\beta)} u(x^\alpha)v(x^\alpha)dV^{(4)}(x^\alpha) , \end{aligned} \tag{4.10}$$

where both $u(x^\alpha)$ and $v(x^\alpha)$ belong to $L_2(\beta)$. We then form the Hilbert space $\mathcal{W}_{34+N}(\beta)$ of ordered (34+N)-tuples of $L_2(\beta)$-functions

$$\underset{\approx}{\omega}(x^\alpha) \equiv \{\theta, x^\beta_K, \theta^K, \mathcal{E}^K, \mathcal{B}^K, \hat{\mathcal{E}}^K, \hat{\mathcal{B}}^K, u_\gamma, \Pi_\Lambda\} . \tag{4.11}$$

Now, we have to make specific assumptions concerning the functional form of the Helmholtz free energy Ψ . We assume that

$$\Psi = \psi(\underset{\approx}{\omega}(x^\alpha), K_\Sigma(x^\alpha)) \;, \qquad \Sigma = 1,\ldots,M \qquad (4.12)$$

where

$$K_\Sigma(x^\alpha) = \int_{(\beta)} g_\Sigma(x^\alpha, z^\alpha; \underset{\approx}{\omega}(x^\alpha), \underset{\approx}{\omega}(z^\alpha))\, dV^{(4)}(z^\alpha) \;. \qquad (4.13)$$

Equations (4.12) and (4.13) indicate that Ψ is a scalar valued function and functional of $\underset{\approx}{\omega}(x^\alpha)$. Thus, we have assumed that Ψ is determined from knowledge of a finite number of functions and functionals of the space-time variables x^α.

Application of the operator D to Ψ gives us

$$\begin{aligned} D\Psi = {} & \frac{\partial}{\partial\theta}(\psi+\overset{*}{\psi})D\theta + \frac{\partial}{\partial x^\beta_K}(\psi+\overset{*}{\psi})Dx^\beta_K \\ & + \frac{\partial}{\partial\theta^K}(\psi+\overset{*}{\psi})D\theta^K + \frac{\partial}{\partial\mathcal{E}^K}(\psi+\overset{*}{\psi})D\mathcal{E}^K \\ & + \frac{\partial}{\partial\mathcal{B}^K}(\psi+\overset{*}{\psi})D\mathcal{B}^K + \frac{\partial}{\partial\hat{\mathcal{E}}^K}(\psi+\overset{*}{\psi})D\hat{\mathcal{E}}^K \\ & + \frac{\partial}{\partial\hat{\mathcal{B}}^K}(\psi+\overset{*}{\psi})D\hat{\mathcal{B}}^K + \frac{\partial}{\partial u_\gamma}(\psi+\overset{*}{\psi})Du_\gamma \\ & + \frac{\partial}{\partial\Pi_\Lambda}(\psi+\overset{*}{\psi})D\Pi_\Lambda + \Phi \;, \end{aligned} \qquad (4.14)$$

where Φ has the property (Edelen [15], Edelen *et al.* [12])

$$\int_{(\beta)} \Phi \, dV^{(4)}(x^\alpha) \equiv 0 \;, \qquad (4.15)$$

and

$$\overset{*}{\psi} = \int_{(\beta)} \frac{\partial\psi}{\partial K_\Sigma} (z^\alpha) \; \overset{*}{g}_\Sigma dV^{(4)}(z^\alpha)$$

$$\overset{*}{g}_\Sigma = g_\Sigma(z^\alpha, x^\alpha; \underset{\approx}{\omega}(z^\alpha), \underset{\approx}{\omega}(x^\alpha)) \tag{4.16}$$

$$g_\Sigma = g_\Sigma(x^\alpha, z^\alpha; \underset{\approx}{\omega}(x^\alpha), \underset{\approx}{\omega}(z^\alpha)) \ .$$

In general, we use the following notation

$$\frac{\partial}{\partial \underset{\approx}{\omega}} (\psi + \overset{*}{\psi}) \equiv \tilde{\delta}_{\underset{\approx}{\omega}} \psi \ , \tag{4.17}$$

so that $\tilde{\delta}_{\underset{\approx}{\omega}}$ is the standard Gateaux derivative operator. Substituting (4.17) into (4.14), we see that

$$\begin{aligned} D\Psi = {} & (\tilde{\delta}_\theta \psi) D\theta + (\tilde{\delta}_{x_K^\beta} \psi) D x_K^\beta + (\delta_{\theta^K} \psi) D\theta^K \\ & + (\tilde{\delta}_{\mathcal{E}^K} \psi) D\mathcal{E}^K + (\tilde{\delta}_{\mathcal{B}^K} \psi) D\mathcal{B}^K + (\tilde{\delta}_{\hat{\mathcal{E}}^K} \psi) D\hat{\mathcal{E}}^K \\ & + (\tilde{\delta}_{\hat{\mathcal{B}}^K} \psi) D\hat{\mathcal{B}}^K + (\tilde{\delta}_{u_\gamma} \psi) D u_\gamma + (\tilde{\delta}_{\Pi_\Lambda} \psi) D\Pi_\Lambda \\ & + \Phi \ . \end{aligned} \tag{4.18}$$

Applying the operator D to (4.7) and (4.9), we see that (3.50) implies

$$D x_K^\alpha = \overset{*}{u}{}^\alpha_\beta x_K^\beta + u^\alpha x_K^\beta D u_\beta \ , \tag{4.19}$$

$$D X^K_{,\alpha} = -u^\beta_{,\alpha} X^K_{,\alpha} \ . \tag{4.20}$$

Applying the operator D to (4.4), using (3.60) and (4.2), and the property (4.20), we show that

$$D(\mathcal{E}^K, \mathcal{B}^K, \hat{\mathcal{E}}^K, \hat{\mathcal{B}}^K, \theta^K) = X^K_{,\alpha}\bar{D}(\mathcal{E}^\alpha, \mathcal{B}^\alpha, \hat{\mathcal{E}}^\alpha, \hat{\mathcal{B}}^\alpha, \overset{*}{\theta}{}^\alpha) \ . \tag{4.21}$$

If we define the tensor $\mathcal{T}^{\alpha\nu}$ by

$$\mathcal{T}^{\alpha\nu} \equiv (\tilde{\delta}_{x^\beta_K}\psi)x^\alpha_K\gamma^{\beta\nu} \ , \tag{4.22}$$

(4.19) and (4.22) show that

$$\begin{aligned}(\tilde{\delta}_{x^\beta_K}\psi)Dx^\beta_K &= \mathcal{T}^{\alpha\beta}\overset{*}{u}_{\alpha\beta} + (\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K Du_\alpha \\ &= \mathcal{T}^{(\alpha\beta)}\overset{*}{u}_{(\alpha\beta)} + \mathcal{T}^{[\alpha\beta]}\overset{*}{u}_{[\alpha\beta]} + (\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K Du_\alpha \\ &= \mathcal{T}^{(\alpha\beta)}\overset{*}{d}_{\alpha\beta} + \mathcal{T}^{[\alpha\beta]}\overset{*}{\omega}_{\alpha\beta} + (\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K Du_\alpha \ .\end{aligned} \tag{4.23}$$

From the definition (4.2), we see that

$$\bar{D}\overset{*}{\theta}{}^\beta = D\overset{*}{\theta}{}^\beta - \overset{*}{\theta}{}^\alpha u^\beta_{,\alpha} \ . \tag{4.24}$$

Thus, substituting (4.21), (4.23) and (4.24) into (4.18), we obtain

$$\begin{aligned}D\Psi = {} & (\tilde{\delta}_\theta\psi)D\theta + \mathcal{T}^{(\alpha\beta)}\overset{*}{d}_{\alpha\beta} + \mathcal{T}^{[\alpha\beta]}\overset{*}{\omega}_{\alpha\beta} + (\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K Du_\alpha \\ & + (\tilde{\delta}_{\theta^K}\psi)X^K_{,\alpha}D\overset{*}{\theta}{}^\alpha + (\tilde{\delta}_{\theta^K}\psi)X^K_{,\beta}u^\beta_{,\alpha}\overset{*}{\theta}{}^\alpha \\ & + (\tilde{\delta}_{\mathcal{E}^K}\psi)X^K_{,\alpha}\bar{D}\mathcal{E}^\alpha + (\tilde{\delta}_{\mathcal{B}^K}\psi)X^K_{,\alpha}\bar{D}\mathcal{B}^\alpha \\ & + (\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha}\bar{D}\hat{\mathcal{E}}^\alpha + (\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)X^K_{,\alpha}\bar{D}\hat{\mathcal{B}}^\alpha \\ & + (\tilde{\delta}_{u_\alpha}\psi)Du_\alpha + (\tilde{\delta}_{\Pi_\Lambda}\psi)D\Pi_\Lambda + \Phi \ .\end{aligned} \tag{4.25}$$

A substitution of (4.25) into (4.3) gives the following working form of the local heat production inequality:

$$
\begin{aligned}
& - n_o[(\tilde{\delta}_\theta\psi)+\eta_{oo}]D\theta + [t^{(\alpha\beta)} - \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} - \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} - \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)} \\
& - \hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)} - n_o\mathcal{T}^{(\alpha\beta)}]\overset{*}{d}_{\alpha\beta} - (n_o\mathcal{T}^{[\alpha\beta]})\overset{*}{\omega}_{\alpha\beta} \\
& - [n_o(\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K + n_o(\tilde{\delta}_{u_\alpha}\psi)]Du_\alpha \\
& - [n_o(\tilde{\delta}_{\theta^K}\psi)X^K_{,\alpha}]D\overset{*}{\theta}{}^\alpha + [n_o(\tilde{\delta}_{\theta^K}\psi)X^K_{,\beta}u^\beta_{,\alpha} - \frac{q_\alpha}{\theta}]\overset{*}{\theta}{}^\alpha \\
& - [n_o(\tilde{\delta}_{\Pi_\Lambda}\psi)]D\Pi_\Lambda - [\mathcal{P}_\alpha + n_o(\tilde{\delta}_{\mathcal{E}^K}\psi)X^K_{,\alpha}]\bar{D}\mathcal{E}^\alpha \\
& - [\mathcal{M}_\alpha + n_o(\tilde{\delta}_{\mathcal{B}^K}\psi)X^K_{,\alpha}]\bar{D}\mathcal{B}^\alpha - [\hat{\mathcal{P}}_\alpha + n_o(\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha}]\bar{D}\hat{\mathcal{E}}^\alpha \\
& - [\hat{\mathcal{M}}_\alpha + n_o(\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)X^K_{,\alpha}]\bar{D}\hat{\mathcal{B}}^\alpha + j^\alpha\mathcal{E}_\alpha + \hat{j}^\alpha\hat{\mathcal{E}}_\alpha \\
& + n_o(\hat{f}_{(mo)} + \hat{S}_o) + \Phi \geq 0 \; . \qquad (4.26)
\end{aligned}
$$

Let us form the space $\mathcal{Y}_{53+N}(\beta)$ of ordered (53+N)-tuples of $L_2(\beta)$-functions which includes the elements

$$
\begin{aligned}
\underset{\approx}{y} \equiv \{ & D\theta, \overset{*}{d}_{\alpha\beta}, \overset{*}{\omega}_{\alpha\beta}, Du_\alpha, D\overset{*}{\theta}{}^\alpha, \overset{*}{\theta}{}^\alpha, D\Pi_\Lambda, \\
& \bar{D}\mathcal{E}^\alpha, \bar{D}\mathcal{B}^\alpha, \bar{D}\hat{\mathcal{E}}^\alpha, \bar{D}\hat{\mathcal{B}}^\alpha, \mathcal{E}_\alpha, \hat{\mathcal{E}}_\alpha\} \; . \qquad (4.27)
\end{aligned}
$$

We also define the operator $\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega})$ from $\mathcal{Y}_{53+N}(\beta)\times \mathcal{W}_{34+N}(\beta)$ into $\mathcal{Y}_{53+N}(\beta)$ by

$$\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega}) \equiv \{-n_o[(\tilde{\delta}_\theta\psi)+\eta_{oo}],\ [t^{(\alpha\beta)}-\mathcal{P}^{(\alpha}\mathcal{E}^{\beta)}-\mathcal{M}^{(\alpha}\mathcal{B}^{\beta)}-\hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)}$$

$$-\hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)}-n_o\mathcal{J}^{(\alpha\beta)}],\ -(n_o\mathcal{J}^{[\alpha\beta]}),\ -[n_o(\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K$$

$$+n_o(\tilde{\delta}_{u_\alpha}\psi)],\ -n_o(\tilde{\delta}_{\theta^K}\psi)X^K_{,\alpha},\ [n_o(\tilde{\delta}_{\theta^K}\psi)X^K_{,\beta}u^\beta_{,\alpha}-\frac{q_\alpha}{\theta}],$$

$$-n_o(\tilde{\delta}_{\Pi_\Lambda}\psi),\ -[\mathcal{P}_\alpha+n_o(\tilde{\delta}_{\mathcal{E}^K}\psi)X^K_{,\alpha}],$$

$$-[\mathcal{M}_\alpha+n_o(\tilde{\delta}_{\mathcal{B}^K}\psi)X^K_{,\alpha}],\ -[\hat{\mathcal{P}}_\alpha+n_o(\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha}],$$

$$-[\hat{\mathcal{M}}_\alpha+n_o(\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)X^K_{,\alpha}],\ j^\alpha,\ \hat{j}^\alpha\}\ . \tag{4.28}$$

Integration of the inequality (4.26) over the whole four-dimensional material tube (β) and use of the zero means properties (3.10), (3.21) and (4.15), we obtain the fundamental statement that the total internal production of heat of any material tube (β) in space-time is non-negative:

$$\int_{(\beta)}\{-n_o[(\tilde{\delta}_\theta\psi)+\eta_{oo}]D\theta+[t^{(\alpha\beta)}-\mathcal{P}^{(\alpha}\mathcal{E}^{\beta)}-\mathcal{M}^{(\alpha}\mathcal{B}^{\beta)}-\hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)}$$

$$-\hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)}-n_o\mathcal{J}^{(\alpha\beta)}]\overset{*}{d}_{\alpha\beta}-(n_o\mathcal{J}^{[\alpha\beta]})\overset{*}{\omega}_{\alpha\beta}$$

$$-[n_o(\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K+n_o(\tilde{\delta}_{u_\alpha}\psi)]Du_\alpha$$

$$-[n_o(\tilde{\delta}_{\theta^K}\psi)X^K_{,\alpha}]D\overset{*}{\theta}{}^\alpha+[n_o(\tilde{\delta}_{\theta^K}\psi)X^K_{,\beta}u^\beta_{,\alpha}-\frac{q_\alpha}{\theta}]\overset{*}{\theta}{}^\alpha$$

$$-[n_o(\tilde{\delta}_{\Pi_\Lambda}\psi)]D\Pi_\Lambda-[\mathcal{P}_\alpha+n_o(\tilde{\delta}_{\mathcal{E}^K}\psi)X^K_{,\alpha}]\bar{D}\mathcal{E}^\alpha$$

$$-[\mathcal{M}_\alpha+n_o(\delta_{\mathcal{B}^K}\psi)X^K_{,\alpha}]\bar{D}\mathcal{B}^\alpha-[\hat{\mathcal{P}}_\alpha+n_o(\delta_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha}]\bar{D}\hat{\mathcal{E}}^\alpha$$

$$- [\hat{\mathfrak{M}}_\alpha + n_o(\tilde{\delta}_{\hat{\mathcal{B}}K}\psi)X^K_{,\alpha}]\bar{D}\hat{\mathcal{B}}^\alpha + j^\alpha\mathcal{E}_\alpha + \hat{j}^\alpha\hat{\mathcal{E}}_\alpha\}dV^{(4)} \geq 0 \;. \tag{4.29}$$

This inequality is equivalent to the functional inequality

$$\langle \underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega}),\underset{\approx}{y}\rangle_{53+N} \geq 0 \;, \tag{4.30}$$

where $\langle\ ,\ \rangle_{53+N}$ is the inner product in $\mathcal{Y}_{53+N}(\beta)$.

Clearly, the specification of a specific form for the left-hand side of (4.28) is equivalent to assigning constitutive relations for the quantities on the right-hand side of (4.28). In fact, the assumption that $\underset{\approx}{J} = \underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega})$ is equivalent to the constitutive assumption that the quantities on the right-hand side of (4.28) depend on the quantities $(\underset{\approx}{y};\underset{\approx}{\omega})$ in view of (4.27) and (4.11).

5. Complete Solutions of the Global Heat Production Inequality for Electromagnetic Interactions with Matter

A general decomposition theorem for vector valued functions (Edelen [16]) has been used in several papers (Edelen [17], [18], [19], [20]) to obtain intrinsically dissipative solutions of the Clausius-Duhem inequality for local theories. We shall obtain similar intrinsically dissipative solutions of the global heat production inequality in this section.

Proofs of the decomposition theorems for operator on function space and solutions of the functional inequalities required in the analysis of irreversible processes in nonlocal continuum theories have been obtained by Edelen [21]. We use the notations, conventions and definitions, which are given by Vainberg [5] and Edelen [4], without further note.

Theorem 1. ([4]) Suppose the following conditions are satisfied:

D_1. The operator $\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega})$ is a mapping from $\mathcal{Y}_{53+N}(\beta)\times\mathcal{W}_{34+N}(\beta)$ into $\mathcal{Y}_{53+N}(\beta)$.

D_2. $\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega})$ has a linear Gateaux differential $\tilde{\delta}_{\underset{\approx}{y}}\underset{\approx}{J}((\underset{\approx}{y},\underset{\approx}{h});\underset{\approx}{\omega})$ at every point of a ball $\mathcal{S}:||\underset{\approx}{y}-\underset{\approx}{y}_o|| < \gamma$, for all $h\varepsilon\mathcal{S}$.

D_3. The functional $\langle \tilde{\delta}_{\underset{\approx}{y}} \underset{\approx}{J}((\underset{\approx}{y},\underset{\approx}{h}_1);\underset{\approx}{\omega}),\underset{\approx}{h}_2\rangle_{53+N}$ is continuous in $\underset{\approx}{y}$ at every point of $\mathcal{D}$, for all $\underset{\approx}{\omega}\varepsilon\mathcal{W}_{34+N}(\beta)$ and all $\underset{\approx}{h}_1,\underset{\approx}{h}_2\varepsilon\mathcal{Y}_{53+N}(\beta)$.

There exists a unique functional $\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega})$ on $\mathcal{D}$, for given $\underset{\approx}{\phi}(\underset{\approx}{y}_o;\underset{\approx}{\omega}) = \underset{\approx}{\phi}_o(\underset{\approx}{\omega})$ and a unique continuous mapping $\underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega})$ from $\mathcal{Y}_{53+N}(\beta)\times\mathcal{W}_{34+N}(\beta)$ into $\mathcal{Y}_{53+N}(\beta)$, with the properties

$$\langle \underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega}),\underset{\approx}{y}-\underset{\approx}{y}_o\rangle = 0 \ , \quad \langle \underset{\approx}{U}(\underset{\approx}{y}_o;\underset{\approx}{\omega}),\underset{\approx}{h}\rangle = 0 \ , \tag{5.1}$$

such that

$$\langle \underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega}),\underset{\approx}{h}\rangle = \langle \tilde{\delta}_{\underset{\approx}{y}}\phi(\underset{\approx}{y};\underset{\approx}{\omega}),\underset{\approx}{h}\rangle + \langle \underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega}),\underset{\approx}{h}\rangle \ , \tag{5.2}$$

for every $\underset{\approx}{y}\varepsilon\mathcal{D}$ and every $\underset{\approx}{h}\varepsilon\mathcal{Y}_{53+N}$. The mapping $\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega})$ and $\underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega})$ are given by

$$\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega}) = \underset{\approx}{\phi}_o(\underset{\approx}{\omega}) + \int_o^1 \langle \underset{\approx}{J}(\underset{\approx}{y}_o+\lambda(\underset{\approx}{y}-\underset{\approx}{y}_o);\underset{\approx}{\omega}),\underset{\approx}{y}-\underset{\approx}{y}_o\rangle d\lambda \ , \tag{5.3}$$

$$\underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega}) = \int_o^1 \lambda\{\tilde{\delta}_{\underset{\approx}{y}}\underset{\approx}{J}((\underset{\approx}{y}_o+\lambda(\underset{\approx}{y}-\underset{\approx}{y}_o),\underset{\approx}{y}-\underset{\approx}{y}_o);\underset{\approx}{\omega}) - \tilde{\overset{*}{\delta}}_{\underset{\approx}{y}}\underset{\approx}{J}((\underset{\approx}{y}_o+\lambda(\underset{\approx}{y}-\underset{\approx}{y}_o), \underset{\approx}{y}-\underset{\approx}{y}_o);\underset{\approx}{\omega})\}d\lambda \tag{5.4}$$

where $\tilde{\overset{*}{\delta}}_{\underset{\approx}{y}}\underset{\approx}{J}((\underset{\approx}{y},\underset{\approx}{h});\underset{\approx}{\omega})$ is defined by

$$\langle \tilde{\delta}_{\underset{\approx}{y}}\underset{\approx}{J}((\underset{\approx}{y},\underset{\approx}{h}_1);\underset{\approx}{\omega}),\underset{\approx}{h}_2\rangle = \langle \tilde{\overset{*}{\delta}}_{\underset{\approx}{y}}\underset{\approx}{J}((\underset{\approx}{y},\underset{\approx}{h}_2);\underset{\approx}{\omega}),\underset{\approx}{h}_1\rangle \ . \tag{5.5}$$

<u>Definition 1</u>. An operator $\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega})$ is said to have the D-property if it satisfied the conditions D_1 through D_3 of Theorem 1. Thus, every operator $\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega})$ with the D-property admits the unique

decomposition

$$J(y;\omega) = \tilde{\delta}_{y}\phi(y;\omega) + U(y;\omega) , \tag{5.6}$$

where $U(y;\omega)$ satisfies the conditions (5.1).

We use 0 to denote the natural elements of spaces $\mathcal{W}_{34+N}(\beta)$ and $\mathcal{Y}_{53+N}(\beta)$ $(||0||=0)$, set $y_o=0$ in Theorem 1. We replace y by an arbitrary element $\overset{*}{y}$ of $\mathcal{Y}_{53+N}(\beta)$, solve

$$\langle J(\overset{*}{y};\omega), \overset{*}{y}\rangle_{53+N} \geq 0 \tag{5.7}$$

on $\mathcal{Y}_{53+N}(\beta)$ for fixed but independent ω , and then evaluate the $J(\overset{*}{y};\omega)$ so obtained for $\overset{*}{y}=y$. We therefore give the following definition.

<u>Definition 2</u>. The collection of all admissible preconstitutive relations consists of all operators $J(\overset{*}{y};\omega)$ with the D-property which satisfy

$$\langle J(\overset{*}{y};\omega), \overset{*}{y}\rangle_{53+N} \geq 0$$

on $\mathcal{Y}_{53+N}(\beta)\times\mathcal{W}_{34+N}(\beta)$, where the elements of $\mathcal{W}_{34+N}$ are considered as independent parameters.

<u>Definition 3</u>. The collection of all admissible constitutive relations consists of all admissible pre-constitutive relations which are then restricted by $\overset{*}{y}=y$.

We see that the operator $\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega})$ in (4.28) are admissible constitutive relations whenever the fundamental inequality (4.30) is satisfied. We thus obtain the following conclusions when Theorem 1 and Definition 2 and 3 are used and we make the identifications $\underset{\approx}{y}_o=\underset{\approx}{0}$.

<u>Theorem 2</u>. All admissible constitutive relations have the form

$$\underset{\approx}{J}(\underset{\approx}{y};\underset{\approx}{\omega}) = \tilde{\delta}_{\underset{\approx}{y}}\phi(\underset{\approx}{y};\underset{\approx}{\omega}) + \underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega}) , \tag{5.8}$$

$$\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega}) = \underset{\approx}{\phi}_o(\underset{\approx}{\omega}) + \int_o^1 \underset{\approx}{P}(\lambda\underset{\approx}{y};\underset{\approx}{\omega}) \frac{d\lambda}{\lambda} , \tag{5.9}$$

where (i) $\underset{\approx}{P}(\underset{\approx}{y};\underset{\approx}{\omega})$ ranges over all functionals on $\mathcal{Y}_{53+N}(\beta)\times\mathcal{W}_{34+N}(\beta)$ which are continuous in $\underset{\approx}{y}$ and $\underset{\approx}{\omega}$ have a continuous Fréchet differential with respect to $\underset{\approx}{y}$ and are such that

$$\underset{\approx}{P}(\underset{\approx}{y};\underset{\approx}{\omega}) \geq 0 , \quad \underset{\approx}{P}(\underset{\approx}{0};\underset{\approx}{\omega}) = 0 , \tag{5.10}$$

and where (ii) $\underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega})$ ranges over all operators with a linear Gateaux differential such that

$$\langle \underset{\approx}{U}(\underset{\approx}{0};\underset{\approx}{\omega}),\underset{\approx}{h}\rangle_{53+N} = 0 , \quad \langle \underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega}),\underset{\approx}{y}\rangle_{53+N} = 0 . \tag{5.11}$$

The following explicit forms of the constitutive relations (5.8) are obtained by use of identifications (4.27) and (4.28):

$$\eta_{oo} = - (\tilde{\delta}_\theta\psi) - \frac{1}{n_o}[(\tilde{\delta}_{D\theta}\phi)+\underset{\approx}{U}_o] , \tag{5.12}$$

$$t^{(\alpha\beta)} = \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} + \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)} + \hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)} + n_o\mathcal{J}^{(\alpha\beta)}$$

$$+ [(\tilde{\delta}_{\overset{*}{d}_{\alpha\beta}}\phi) + \underset{\approx}{U}^{(\alpha\beta)}] \; , \qquad (5.13)$$

$$n_o\mathcal{J}^{[\alpha\beta]} = -[(\tilde{\delta}_{\overset{*}{\omega}_{\alpha\beta}}\phi) + \underset{\approx}{U}^{[\alpha\beta]}] \; , \qquad (5.14)$$

$$-[n_o(\tilde{\delta}_{x^{\beta}_{K}}\psi)u^{\beta}x^{\alpha}_{K} + n_o(\tilde{\delta}_{\mu_\alpha}\psi) = (\tilde{\delta}_{D\mu_\alpha}\phi) + \underset{\approx}{U}^{\alpha}_{o} \; , \qquad (5.15)$$

$$-n_o(\tilde{\delta}_{\theta^K}\psi)X^{K}_{,\alpha} = (\tilde{\delta}_{D\overset{*}{\theta}^{\alpha}}\phi) + \underset{\approx}{U}_{1\alpha} \; , \qquad (5.16)$$

$$q_\alpha = -\theta[(\tilde{\delta}_{\overset{*}{\theta}^{\alpha}}\phi) + \underset{\approx}{U}_{2\alpha} - n_o(\tilde{\delta}_{\theta^K}\psi)X^{K}_{,\beta}u^{\beta}_{,\alpha}] \; , \qquad (5.17)$$

$$-n_o(\tilde{\delta}_{\Pi_\Lambda}\psi) = (\tilde{\delta}_{D\Pi_\Lambda}\phi) + \underset{\approx}{U}^{\Lambda} \; , \qquad (5.18)$$

$$\mathcal{P}_\alpha = -(\tilde{\delta}_{\bar{D}\mathcal{E}^{\alpha}}\phi) - \underset{\approx}{U}_{3\alpha} - n_o(\tilde{\delta}_{\mathcal{E}^K}\psi)X^{K}_{,\alpha} \; , \qquad (5.19)$$

$$\mathcal{M}_\alpha = -(\tilde{\delta}_{\bar{D}\mathcal{B}^{\alpha}}\phi) - \underset{\approx}{U}_{4\alpha} - n_o(\tilde{\delta}_{\mathcal{B}^K}\psi)X^{K}_{,\alpha} \; , \qquad (5.20)$$

$$\hat{\mathcal{P}}_\alpha = -(\tilde{\delta}_{\bar{D}\hat{\mathcal{E}}^{\alpha}}\phi) - \underset{\approx}{U}_{5\alpha} - n_o(\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)X^{K}_{,\alpha} \; , \qquad (5.21)$$

$$\hat{\mathcal{M}}_\alpha = -(\tilde{\delta}_{\bar{D}\hat{\mathcal{B}}^{\alpha}}\phi) - \underset{\approx}{U}_{6\alpha} - n_o(\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)X^{K}_{,\alpha} \; , \qquad (5.22)$$

$$j^{\alpha} = (\tilde{\delta}_{\mathcal{E}^{\alpha}}\phi) + \underset{\approx}{U}_{7\alpha} \; , \qquad (5.23)$$

$$\hat{j}^{\alpha} = (\tilde{\delta}_{\hat{\mathcal{E}}^{\alpha}}\phi) + \underset{\approx}{U}_{8\alpha} \; , \qquad (5.24)$$

where the $\underset{\approx}{U}$'s are such that

$$\left\langle \underset{\approx}{U}_o(\underset{\sim}{y};\underset{\approx}{\omega}), D\theta \right\rangle + \left\langle \underset{\approx}{U}^{(\alpha\beta)}(\underset{\sim}{y};\underset{\approx}{\omega}), \overset{*}{d}_{\alpha\beta} \right\rangle + \left\langle \underset{\approx}{U}^{[\alpha\beta]}(\underset{\sim}{y};\underset{\approx}{\omega}), \overset{*}{\omega}_{\alpha\beta} \right\rangle$$

$$+ \left\langle \underset{\approx}{U}^{\alpha}_{o}(\underset{\sim}{y};\underset{\approx}{\omega}), Du_{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{1\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), D\overset{*}{\theta}{}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{2\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), \overset{*}{\theta}{}^{\alpha} \right\rangle$$

$$+ \left\langle \underset{\approx}{U}^{\Lambda}(\underset{\sim}{y};\underset{\approx}{\omega}), D\Pi_{\Lambda} \right\rangle + \left\langle \underset{\approx}{U}_{3\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), \tilde{D}\mathcal{E}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{4\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), \tilde{D}\mathcal{B}^{\alpha} \right\rangle$$

$$+ \left\langle \underset{\approx}{U}_{5\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), \tilde{D}\hat{\mathcal{E}}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{6\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), \tilde{D}\hat{\mathcal{B}}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{7\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), \mathcal{E}^{\alpha} \right\rangle$$

$$+ \left\langle \underset{\approx}{U}_{8\alpha}(\underset{\sim}{y};\underset{\approx}{\omega}), \hat{\mathcal{E}}^{\alpha} \right\rangle = 0 , \tag{5.25}$$

$$\underset{\approx}{U}_o(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}^{(\alpha\beta)}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}^{[\alpha\beta]}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}^{\alpha}_{o}(\underset{\approx}{0};\underset{\approx}{\omega})$$

$$= \underset{\approx}{U}_{1\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{2\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}^{\Lambda}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{3\alpha}(\underset{\approx}{0};\underset{\approx}{\omega})$$

$$= \underset{\approx}{U}_{4\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{5\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{6\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{7\alpha}(\underset{\approx}{0};\underset{\approx}{\omega})$$

$$= \underset{\approx}{U}_{8\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = 0 , \tag{5.26}$$

from (5.11); and $\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega})$ is given by (5.9) subject to conditions (5.10), on the functional $\underset{\approx}{P}(\underset{\approx}{y};\underset{\approx}{\omega})$.

Now, we give three implications as follows:

(I) If we assume

$$\underset{\approx}{U}^{[\alpha\beta]} = \underset{\approx}{U}^{\alpha}_{o} = \underset{\approx}{U}^{\alpha}_{1} = \underset{\approx}{U}^{\Lambda} = 0 , \tag{5.27}$$

then we obtain

$$n_o \mathcal{J}^{[\alpha\beta]} = -(\tilde{\delta}_{\overset{*}{\omega}_{\alpha\beta}} \phi) , \tag{5.28}$$

$$-[n_o(\tilde{\delta}_{x^{\beta}_{K}}\psi)u^{\beta}x^{\alpha}_{K} + n_o(\tilde{\delta}_{\mu_{\alpha}}\psi)] = \tilde{\delta}_{D\mu_{\alpha}}\phi , \tag{5.29}$$

$$-n_o(\tilde{\delta}_{\theta^K}\psi)X^K_{,\alpha} = \tilde{\delta}_{D\overset{*}{\theta}^\alpha}\phi \ , \tag{5.30}$$

$$-n_o(\tilde{\delta}_{\Pi_\Lambda}\psi) = \tilde{\delta}_{D\Pi_\Lambda}\phi \ . \tag{5.31}$$

From (4.22), (5.28), (5.29), (5.30) and (5.31), we see that $\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega})$ is functional independent with respect to $\overset{*}{\omega}_{\alpha\beta}$, Du_α, $D\overset{*}{\theta}^\alpha$ and $D\Pi_\Lambda$ (i.e. $\tilde{\delta}_{D\overset{*}{\omega}_{\alpha\beta}}\phi = \tilde{\delta}_{D\mu_\alpha}\phi = \tilde{\delta}_{D\overset{*}{\theta}^\alpha}\phi = \tilde{\delta}_{D\Pi_\Lambda}\phi = 0$, since linear functional dependence would violate the condition $\underset{\approx}{P}(\underset{\approx}{y};\underset{\approx}{\omega}) \geq 0$).

Accordingly, we have

$$n_o\mathcal{T}^{[\alpha\beta]} \equiv 0 \ , \tag{5.32}$$

$$n_o(\tilde{\delta}_{x^\beta_K}\psi)u^\beta x^\alpha_K + n_o(\tilde{\delta}_{\mu_\alpha}\psi) \equiv 0 \ , \tag{5.33}$$

$$\tilde{\delta}_{\theta^K}\psi \equiv 0 \ , \tag{5.34}$$

$$\tilde{\delta}_{\Pi_\Lambda}\psi \equiv 0 \ . \tag{5.35}$$

However, (4.22) and $\mathcal{T}^{[\alpha\beta]} \equiv 0$ show that $[\psi(\underset{\approx}{\omega})+\overset{*}{\psi}(\underset{\approx}{\omega})]$ can functionally depend on x^β_K only through

$$\bar{C}^{1}_{MN} = \gamma_{\mu\nu}x^\mu_M x^\nu_N \ . \tag{5.36}$$

Accordingly, $(\tilde{\delta}_{x^\beta_K}\psi)u^\beta \equiv 0$, and (5.33) yields

$$\tilde{\delta}_{\mu_\alpha}\psi \equiv 0 \ . \tag{5.37}$$

From (5.34), (5.35) and (5.37), we see that $[\psi(\underset{\approx}{\omega})+\overset{*}{\psi}(\underset{\approx}{\omega})]$ is functionally independent of θ^K, Π_Λ and u_α .

A sufficient, but not necessary condition for satisfaction of these requirements is that the Helmholtz free energy function have the form

$$\Psi = \psi(\theta(x^\alpha), \bar{C}^{1}_{MN}(x^\alpha), \mathcal{E}^K(x^\alpha), \mathcal{B}^K(x^\alpha), \hat{\mathcal{E}}^K(x^\alpha), \hat{\mathcal{B}}^K(x^\alpha), K_\Sigma(x^\alpha)),$$
$$\Sigma = 1,\ldots,M \qquad (5.38)$$

where

$$K_\Sigma(x^\alpha) = \int_{(\beta)} g_\Sigma(x^\alpha, z^\alpha; \theta(x^\alpha), \theta(z^\alpha), \bar{C}^{1}_{MN}(x^\alpha), \bar{C}^{1}_{MN}(z^\alpha),$$
$$\mathcal{E}^K(x^\alpha), \mathcal{E}^K(z^\alpha), \mathcal{B}^K(x^\alpha), \mathcal{B}^K(z^\alpha), \hat{\mathcal{E}}^K(x^\alpha), \hat{\mathcal{E}}^K(z^\alpha),$$
$$\hat{\mathcal{B}}^K(x^\alpha), \hat{\mathcal{B}}^K(z^\alpha))dV^{(4)}(z^\alpha) \;, \quad \Sigma = 1,\ldots,M. \qquad (5.39)$$

In this case, (5.12)-(5.26) reduce to

$$\eta_{oo} = -(\tilde{\delta}_\theta\psi) - \frac{1}{n_o}[(\tilde{\delta}_{D\theta}\phi) + \underset{\approx}{U}_o] \;, \qquad (5.40)$$

$$t^{(\alpha\beta)} = \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} + \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)} + \hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)} + n_o\mathcal{T}^{(\alpha\beta)}$$
$$+ [(\tilde{\delta}_{\overset{*}{d}_{\alpha\beta}}\phi) + \underset{\approx}{U}^{(\alpha\beta)}] \;, \qquad (5.41)$$

$$q_\alpha = -\theta[(\tilde{\delta}_{\overset{*}{\theta}\alpha}\phi) + \underset{\approx}{U}_{2\alpha}] \;, \qquad (5.42)$$

$$\mathcal{P}_\alpha = -(\tilde{\delta}_{\bar{D}\mathcal{E}^\alpha}\phi) - \underset{\approx}{U}_{3\alpha} - n_o(\tilde{\delta}_{\mathcal{E}^K}\psi)X^K_{,\alpha} \;, \qquad (5.43)$$

$$\mathcal{M}_\alpha = -(\tilde{\delta}_{\bar{D}\mathcal{B}^\alpha}\phi) - \underset{\approx}{U}_{4\alpha} - n_o(\tilde{\delta}_{\mathcal{B}^K}\psi)X^K_{,\alpha} \;, \qquad (5.44)$$

$$\hat{\mathcal{P}}_\alpha = -(\tilde{\delta}_{\bar{D}\hat{\mathcal{E}}^\alpha}\phi) - \underset{\approx}{U}_{5\alpha} - n_o(\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha} \;, \qquad (5.45)$$

$$\hat{\mathcal{M}}_\alpha = -(\tilde{\delta}_{\bar{D}\hat{\mathcal{B}}^\alpha}\phi) - \underset{\approx}{U}_{6\alpha} - n_o(\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)X^K_{,\alpha} \;, \qquad (5.46)$$

$$j^{\alpha} = (\tilde{\delta}_{\mathcal{E}^{\alpha}}\phi) + \underset{\approx}{U}_{7\alpha} , \tag{5.47}$$

$$\hat{j}^{\alpha} = (\tilde{\delta}_{\hat{\mathcal{E}}^{\alpha}}\phi) + \underset{\approx}{U}_{8\alpha} , \tag{5.48}$$

$$\left\langle \underset{\approx}{U}_{o}(\underset{\approx}{y};\underset{\approx}{\omega}), D\theta \right\rangle + \left\langle \underset{\approx}{U}^{(\alpha\beta)}(\underset{\approx}{y};\underset{\approx}{\omega}), \overset{*}{d}_{\alpha\beta} \right\rangle + \left\langle \underset{\approx}{U}_{2\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}), \overset{*}{\theta}^{\alpha} \right\rangle$$
$$+ \left\langle \underset{\approx}{U}_{3\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}), \bar{D}\mathcal{E}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{4\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}), \bar{D}\mathcal{B}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{5\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}), \bar{D}\hat{\mathcal{E}}^{\alpha} \right\rangle$$
$$+ \left\langle \underset{\approx}{U}_{6\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}), \bar{D}\hat{\mathcal{B}}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{7\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}), \mathcal{E}^{\alpha} \right\rangle + \left\langle \underset{\approx}{U}_{8\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}), \hat{\mathcal{E}}^{\alpha} \right\rangle = 0 , \tag{5.49}$$

$$\underset{\approx}{U}_{o}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}^{(\alpha\beta)}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{2\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{3\alpha}(\underset{\approx}{0};\underset{\approx}{\omega})$$
$$= \underset{\approx}{U}_{4\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{5\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{6\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{7\alpha}(\underset{\approx}{0};\underset{\approx}{\omega})$$
$$= \underset{\approx}{U}_{8\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = 0 . \tag{5.50}$$

(II) If we set all of the $\underset{\approx}{U}$'s equal to zero (so that we obtain the generalized form of Onsager's reciprocity relations (Edelen [4]), we have

$$\eta_{oo} = -(\tilde{\delta}_{\theta}\psi) - \frac{1}{n_o}[(\tilde{\delta}_{D\theta}\phi) , \tag{5.51}$$

$$t^{(\alpha\beta)} = \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} + \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)} + \hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)} + n_o\mathcal{T}^{(\alpha\beta)}$$
$$+ (\tilde{\delta}_{\overset{*}{d}_{\alpha\beta}}\phi) , \tag{5.52}$$

$$q_{\alpha} = -\theta(\tilde{\delta}_{\overset{*}{\theta}^{\alpha}}\phi) , \tag{5.53}$$

$$\mathcal{P}_{\alpha} = -(\tilde{\delta}_{\bar{D}\mathcal{E}^{\alpha}}\phi) - n_o(\tilde{\delta}_{\mathcal{E}^{K}}\psi)X^{K}_{;\alpha} , \tag{5.54}$$

$$\mathcal{M}_{\alpha} = -(\tilde{\delta}_{\bar{D}\mathcal{B}^{\alpha}}\phi) - n_o(\tilde{\delta}_{\mathcal{B}^{K}}\psi)X^{K}_{;\alpha} , \tag{5.55}$$

$$\hat{\mathscr{P}}_\alpha = -(\tilde{\delta}_{\bar{D}\hat{\mathcal{E}}^\alpha}\phi) - n_o(\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha} , \tag{5.56}$$

$$\hat{\mathfrak{M}}_\alpha = -(\tilde{\delta}_{\bar{D}\hat{\mathcal{B}}^\alpha}\phi) - n_o(\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)X^K_{,\alpha} , \tag{5.57}$$

$$j^\alpha = (\tilde{\delta}_{\mathcal{E}^\alpha}\phi) , \tag{5.58}$$

$$\hat{j}^\alpha = (\tilde{\delta}_{\hat{\mathcal{E}}^\alpha}\phi) . \tag{5.59}$$

(III) If we assume $\psi(\underset{\approx}{\omega})$ depends on $\underset{\approx}{\omega}$ but not on functionals of $\underset{\approx}{\omega}$, then $\tilde{\delta}_{\underset{\approx}{\omega}}\psi = \partial_{\underset{\approx}{\omega}}\psi$. Similarly, if $\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega})$ depends on $\underset{\approx}{y}$ and $\underset{\approx}{\omega}$ but not on functionals of $\underset{\approx}{y}$ and $\underset{\approx}{\omega}$, then $\tilde{\delta}_{\underset{\approx}{y}}\phi = \partial_{\underset{\approx}{y}}\phi$. In this event, we put

$$U^{[\alpha\beta]} = U^\alpha_o = U^\alpha_1 = U^\Lambda = 0 , \tag{5.60}$$

and we obtain

$$n_o\mathfrak{J}^{[\alpha\beta]} = \partial_{\overset{*}{\omega}_{\alpha\beta}}\phi , \tag{5.61}$$

$$n_o(\partial_{x^\beta_K}\psi)u^\beta x^\alpha_K + n_o(\partial_{\mu_\alpha}\psi) = -\partial_{D\mu_\alpha}\phi , \tag{5.62}$$

$$-n_o(\partial_{\theta^K}\psi)X^K_{,\alpha} = (\partial_{D\overset{*}{\theta}^\alpha}\phi) , \tag{5.63}$$

$$-n_o(\partial_{\Pi_\Lambda}\psi) = (\partial_{D\Pi_\Lambda}\phi) . \tag{5.64}$$

From (4.22), (5.61), (5.62), (5.63) and (5.64), we see that $\underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega})$ is function independent with respect to $\overset{*}{\omega}_{\alpha\beta}$, Du_α, $D\overset{*}{\theta}^\alpha$ and $D\Pi_\Lambda$ (i.e. $\partial_{\overset{*}{\omega}_{\alpha\beta}}\phi = \partial_{D\mu_\alpha}\phi = \partial_{D\overset{*}{\theta}^\alpha}\phi = \partial_{D\Pi_\Lambda}\phi \equiv 0$), these imply

$$\mathcal{J}^{[\alpha\beta]} \equiv 0 \ , \tag{5.65}$$

$$n_o(\partial_{x_K^\beta}\psi)u^\beta x_K^\alpha + n_o(\partial_{\mu_\alpha}\psi) \equiv 0 \ , \tag{5.66}$$

$$\partial_{\theta^K}\psi \equiv 0 \ , \tag{5.67}$$

$$\partial_{\Pi_\Lambda}\psi \equiv 0 \ . \tag{5.68}$$

However, (4.22) and $\mathcal{J}^{[\alpha\beta]} \equiv 0$ show that $\psi(\underset{\approx}{\omega})$ can depend on x_K^β only through

$$\bar{C}_{MN}^{1} = \gamma_{\mu\nu}x_M^\mu x_N^\nu \ . \tag{5.69}$$

Accordingly, $(\partial_{x_K^\beta}\psi)u^\beta \equiv 0$, and (5.66) yields

$$\partial_{\mu_\alpha}\psi \equiv 0 \ . \tag{5.70}$$

From (5.67), (5.68) and (5.70), it follows that we can only have a Helmholtz free energy function of the form Ψ

$$\Psi = \psi(\theta,\bar{C}_{MN}^{1},\mathcal{E}^K,\mathcal{B}^K,\hat{\mathcal{E}}^K,\hat{\mathcal{B}}^K) \ . \tag{5.71}$$

Under these conditions, the constitutive relations become

$$\eta_{oo} = -(\partial_\theta\psi) - \frac{1}{n_o}[(\partial_{D\theta}\phi)+\underset{\approx}{U}_o] \ , \tag{5.72}$$

$$t^{(\alpha\beta)} = \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} + \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)} + \hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)} + n_o\mathcal{J}^{\alpha\beta}$$
$$+ (\partial_{\overset{*}{d}_{\alpha\beta}}\phi) + \underset{\approx}{U}^{(\alpha\beta)} \ , \tag{5.73}$$

$$q_\alpha = -\theta[(\partial_{\overset{*}{\theta}\alpha}\phi)+\underset{\approx}{U}_{2\alpha}] \ , \tag{5.74}$$

$$\mathcal{P}_\alpha = -(\partial_{\bar{D}\mathcal{E}^\alpha}\phi) - \underset{\approx}{U}_{3\alpha} - n_o(\partial_{\mathcal{E}^K}\psi)X^K_{,\alpha} \ , \tag{5.75}$$

$$\mathcal{M}_\alpha = -(\partial_{\bar{D}\mathcal{B}^\alpha}\phi) - \underset{\approx}{U}_{4\alpha} - n_o(\partial_{\mathcal{B}^K}\psi)X^K_{,\alpha} \ , \tag{5.76}$$

$$\hat{\mathcal{P}}_\alpha = -(\partial_{\bar{D}\hat{\mathcal{E}}^\alpha}\phi) - \underset{\approx}{U}_{5\alpha} - n_o(\partial_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha} \ , \tag{5.77}$$

$$\hat{\mathcal{M}}_\alpha = -(\partial_{\bar{D}\hat{\mathcal{B}}^\alpha}\phi) - \underset{\approx}{U}_{6\alpha} - n_o(\partial_{\hat{\mathcal{B}}^K}\psi)X^K_{,\alpha} \ , \tag{5.78}$$

$$j^\alpha = (\partial_{\mathcal{E}^\alpha}\phi) + \underset{\approx}{U}_{7\alpha} \ , \tag{5.79}$$

$$\hat{j}^\alpha = (\partial_{\hat{\mathcal{E}}^\alpha}\phi) + \underset{\approx}{U}_{8\alpha} \ , \tag{5.80}$$

$$\begin{aligned}&\left\langle \underset{\approx}{U}_o(\underset{\approx}{y};\underset{\approx}{\omega}),D\theta\right\rangle + \left\langle \underset{\approx}{U}^{(\alpha\beta)}(\underset{\approx}{y};\underset{\approx}{\omega}),\overset{*}{d}_{\alpha\beta}\right\rangle + \left\langle \underset{\approx}{U}_{2\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}),\overset{*}{\theta}^\alpha\right\rangle \\ &+ \left\langle \underset{\approx}{U}_{3\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}),\bar{D}\mathcal{E}^\alpha\right\rangle + \left\langle \underset{\approx}{U}_{4\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}),\bar{D}\mathcal{B}^\alpha\right\rangle + \left\langle \underset{\approx}{U}_{5\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}),\bar{D}\hat{\mathcal{E}}^\alpha\right\rangle \\ &+ \left\langle \underset{\approx}{U}_{6\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}),\bar{D}\hat{\mathcal{B}}^\alpha\right\rangle + \left\langle \underset{\approx}{U}_{7\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}),\mathcal{E}^\alpha\right\rangle + \left\langle \underset{\approx}{U}_{8\alpha}(\underset{\approx}{y};\underset{\approx}{\omega}),\hat{\mathcal{E}}^\alpha\right\rangle = 0 \ ,\end{aligned} \tag{5.81}$$

$$\begin{aligned}&\underset{\approx}{U}_o(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}^{(\alpha\beta)}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{2\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{3\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) \\ &= \underset{\approx}{U}_{4\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{5\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{6\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{U}_{7\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) \\ &= \underset{\approx}{U}_{8\alpha}(\underset{\approx}{0};\underset{\approx}{\omega}) = 0 \ .\end{aligned} \tag{5.82}$$

These results are clearly dynamics since they give dependences on $D\theta$, $\overset{*}{d}_{\alpha\beta}$, $\overset{*}{\theta}^\alpha$, $\bar{D}\mathcal{E}^\alpha$, $\bar{D}\mathcal{B}^\alpha$, $\bar{D}\hat{\mathcal{E}}^\alpha$, $\bar{D}\hat{\mathcal{B}}^\alpha$, $\mathcal{E}^\alpha$ and $\hat{\mathcal{E}}^\alpha$. It is also tempting at this point to set all of the remaining $\underset{\approx}{U}$'s equal to zero so that we obtain the generalized form of Onsager's reciprocity relations. We would then have the constitutive relations

$$\eta_{oo} = -(\partial_\theta \psi) - \frac{1}{n_o}(\partial_{D\theta}\phi) \ , \tag{5.83}$$

$$t^{(\alpha\beta)} = \mathcal{P}^{(\alpha}\mathcal{E}^{\beta)} + \mathcal{M}^{(\alpha}\mathcal{B}^{\beta)} + \hat{\mathcal{P}}^{(\alpha}\hat{\mathcal{E}}^{\beta)} + \hat{\mathcal{M}}^{(\alpha}\hat{\mathcal{B}}^{\beta)} + n_o\mathcal{T}^{\alpha\beta}$$
$$+ \partial_{\overset{*}{d}_{\alpha\beta}}\phi \ , \tag{5.84}$$

$$q_\alpha = -\theta(\partial_{\overset{*}{\theta}{}^\alpha}\phi) \ , \tag{5.85}$$

$$\mathcal{P}_\alpha = -(\partial_{\bar{D}\mathcal{E}^\alpha}\phi) - n_o(\partial_{\mathcal{E}^K}\psi)X^K_{,\alpha} \ , \tag{5.86}$$

$$\mathcal{M}_\alpha = -(\partial_{\bar{D}\mathcal{B}^\alpha}\phi) - n_o(\partial_{\mathcal{B}^K}\psi)X^K_{,\alpha} \ , \tag{5.87}$$

$$\hat{\mathcal{P}}_\alpha = -(\partial_{\bar{D}\hat{\mathcal{E}}^\alpha}\phi) - n_o(\partial_{\hat{\mathcal{E}}^K}\psi)X^K_{,\alpha} \ , \tag{5.88}$$

$$\hat{\mathcal{M}}_\alpha = -(\partial_{\bar{D}\hat{\mathcal{B}}^\alpha}\phi) - n_o(\partial_{\hat{\mathcal{B}}^K}\psi)X^K_{,\alpha} \ , \tag{5.89}$$

$$j^\alpha = \partial_{\mathcal{E}^\alpha}\phi \ , \tag{5.90}$$

$$\hat{j}^\alpha = \partial_{\hat{\mathcal{E}}^\alpha}\phi \ . \tag{5.91}$$

We also note that all of these systems of constitutive relations agree in thermodynamic equilibrium states since $\underset{\approx}{y}=\underset{\approx}{0}$ for such states and hence $\min\limits_{\underset{\approx}{y}\varepsilon} \underset{\approx}{\phi}(\underset{\approx}{y};\underset{\approx}{\omega}) = \underset{\approx}{\phi}(\underset{\approx}{0};\underset{\approx}{\omega})$, $\underset{\approx}{U}(\underset{\approx}{0};\underset{\approx}{\omega}) = \underset{\approx}{0}$.

6. Problems Which Remain to be Solved

The basic theory of nonlocal electromagnetic interactions with matter from the relativistic point of view is now complete. However, there are three remaining representation problems which must be resolved before a full theory is obtained.

The first of these problems is that of obtaining representation for the localization residuals that satisfy the required zero mean conditions. It seems that satisfaction of the zero mean condition for $\hat{f}^{\alpha}_{(m)}$ can not be obtained from the implications of invariance of the Helmholtz free energy Ψ under Lorentz transformation and the admissible constitutive relations. However, Lorentz invariance of Ψ does imply satisfaction of the parts of zero mean conditions for $\hat{\ell}^{\alpha\beta}$, $\hat{\phi}^{\alpha\beta}$ and $\hat{\pi}^{\alpha\beta}$ that contain functional derivatives of ψ. To make the discussion specific, we simply consider the Helmholtz free energy Ψ of the form

$$\Psi = \psi(\underset{\approx}{\omega}'(x^{\alpha}), K'_{\Sigma}(x^{\alpha})) \quad , \qquad \Sigma = 1,\ldots,M \ , \tag{6.1}$$

where

$$\underset{\approx}{\omega}'(x^{\alpha}) = \{\theta(x^{\alpha}), \mathcal{E}^{K}(x^{\alpha}), \mathcal{B}^{K}(x^{\alpha}), \hat{\mathcal{E}}^{K}(x^{\alpha}), \hat{\mathcal{B}}^{K}(x^{\alpha})\} \ , \tag{6.2}$$

$$K'_{\Sigma}(x^{\alpha}) = \int_{(\beta)} g_{\Sigma}(x^{\alpha}, z^{\alpha}; \underset{\approx}{\omega}'(x^{\alpha}), \underset{\approx}{\omega}'(z^{\alpha}))dV^{(4)}(z^{\alpha}) \ ,$$

$$\Sigma = 1,\ldots,M \ . \tag{6.3}$$

We denote infinitesimal Lorentz transformation by (Edelen [22]):

$$x'^{\alpha} = x^{\alpha} + (a^{\alpha}+\omega^{\alpha\beta}\gamma_{\beta\gamma}x^{\gamma})e + 0(e) \; . \tag{6.4}$$

Here, e denotes an infinitesimal parameter, the a^{α}'s are constants (infinitesimal generators of the translation subgroup), and the quantity $(\omega^{\alpha\beta})$ is the tensor that generates the proper Lorentz group of 4-space rotations. In terms of 3-dimensional relative velocity $\underset{\sim}{V}$ between the frames (x'^{α}) and (x^{α}) and the 3-dimensional rotation vector $\underset{\sim}{\omega}$, we have

$$\omega^{\alpha\beta} = \begin{pmatrix} 0 & \omega_z & -\omega_y & V_x \\ -\omega_z & 0 & \omega_x & V_y \\ \omega_y & -\omega_x & 0 & V_z \\ V_x & V_y & V_z & 0 \end{pmatrix} , \tag{6.5}$$

and it is obvious that there are only six independent elements in (6.5). We define

$$\omega^{\alpha\beta} \equiv \underset{(i)}{\omega^{\alpha\beta}} \overset{(i)}{b} , \quad (i) = 1,\ldots,6 , \tag{6.6}$$

where $\overset{(i)}{b}$ are the 6 independent parameters $(\omega_x, \omega_y, \omega_3, V_x, V_y, V_3)$ that occur in (6.5) and $\underset{(i)}{\omega^{\alpha\beta}}$ are specific numerical matrices that are then obtained directly from (6.5). The definition of $\mathcal{E}^K$, $\mathcal{B}^K$, $\hat{\mathcal{E}}^K$ and $\hat{\mathcal{B}}^K$ give their transformation properties under infinitesimal Lorentz transformation, so that we obtain

$$\delta \xi^K = \underset{(i)}{\omega}^{\alpha\beta} \overset{(i)}{b} (A^K_{\alpha\beta L} \xi^L + B^K_{\alpha\beta L} \mathcal{B}^L) , \qquad (6.7)$$

$$\delta \mathcal{B}^K = \underset{(i)}{\omega}^{\alpha\beta} \overset{(i)}{b} (\bar{A}^K_{\alpha\beta L} \xi^L + \bar{B}^K_{\alpha\beta L} \mathcal{B}^L) , \qquad (6.8)$$

$$\delta \hat{\xi}^K = \underset{(i)}{\omega}^{\alpha\beta} \overset{(i)}{b} (A^K_{\alpha\beta L} \hat{\xi}^L + B^K_{\alpha\beta L} \hat{\mathcal{B}}^L) , \qquad (6.9)$$

$$\delta \hat{\mathcal{B}}^K = \underset{(i)}{\omega}^{\alpha\beta} \overset{(i)}{b} (\bar{A}^K_{\alpha\beta L} \hat{\xi}^L + \bar{B}^K_{\alpha\beta L} \hat{\mathcal{B}}^L) , \qquad (6.10)$$

where $A^K_{\alpha\beta L}$, $B^K_{\alpha\beta L}$, $\bar{A}^K_{\alpha\beta L}$ and $\bar{B}^K_{\alpha\beta L}$ are numerical matrices. Since the Helmholtz free energy Ψ is invariant under infinitesimal Lorentz transformation, we obtain

$$\begin{aligned} 0 = \delta\Psi = {} & (\tilde{\delta}_\theta \psi)\delta\theta + (\tilde{\delta}_{\xi^K}\psi)\delta\xi^K + (\tilde{\delta}_{\mathcal{B}^K}\psi)\delta\mathcal{B}^K \\ & + (\tilde{\delta}_{\hat{\xi}^K}\psi)\delta\hat{\xi}^K + (\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)\delta\hat{\mathcal{B}}^K \\ & + \hat{R}_\theta(\underset{\approx}{\omega}' : \delta\theta) + \hat{R}_{\xi^K}(\underset{\approx}{\omega}' : \delta\xi^K) + \hat{R}_{\mathcal{B}^K}(\underset{\approx}{\omega}' : \delta\mathcal{B}^K) \\ & + \hat{R}_{\hat{\xi}^K}(\underset{\approx}{\omega}' : \delta\hat{\xi}^K) + \hat{R}_{\hat{\mathcal{B}}^K}(\underset{\approx}{\omega}' : \delta\hat{\mathcal{B}}^K) , \end{aligned} \qquad (6.11)$$

where $\hat{R}_\theta(\underset{\approx}{\omega}' : \delta\theta)$, $\hat{R}_{\xi^K}(\underset{\approx}{\omega}' : \delta\xi^K)$, $\hat{R}_{\mathcal{B}^K}(\underset{\approx}{\omega}' : \delta\mathcal{B}^K)$, $\hat{R}_{\hat{\xi}^K}(\underset{\approx}{\omega}' : \delta\hat{\xi}^K)$ and $\hat{R}_{\hat{\mathcal{B}}^K}(\underset{\approx}{\omega}' : \delta\hat{\mathcal{B}}^K)$ are localization residuals which are linear (functionals) of the arguments following the colon (:), that is

$$\left\langle 1,\hat{R}_\theta(\underset{\approx}{\omega}':\delta\theta)\right\rangle = \left\langle 1,\hat{R}_{\mathcal{E}^K}(\underset{\approx}{\omega}':\delta\mathcal{E}^K)\right\rangle$$

$$= \left\langle 1,\hat{R}_{\mathcal{B}^K}(\underset{\approx}{\omega}':\delta\mathcal{B}^K)\right\rangle = \left\langle 1,\hat{R}_{\hat{\mathcal{E}}^K}(\underset{\approx}{\omega}':\delta\hat{\mathcal{E}}^K)\right\rangle$$

$$= \left\langle 1,\hat{R}_{\hat{\mathcal{B}}^K}(\underset{\approx}{\omega}':\delta\hat{\mathcal{B}}^K)\right\rangle = 0 \ . \tag{6.12}$$

Since θ is invariant under infinitesimal Lorentz transformation we further obtain

$$\delta\theta = 0 \ , \quad \hat{R}_\theta(\underset{\approx}{\omega}':\delta\theta) = 0 \ . \tag{6.13}$$

Substituting (6.7), (6.8), (6.9), (6.10) and (6.13) into (6.11), and noting that the results hold for arbitrary $\overset{(i)}{b}$, we obtain

$$\begin{aligned} 0 = {} & \underset{(i)}{\omega}^{\alpha\beta}[(\tilde{\delta}_{\mathcal{E}^K}\psi)(A^K_{\alpha\beta L}\mathcal{E}^L+B^K_{\alpha\beta L}\mathcal{B}^L) \\ & + (\tilde{\delta}_{\mathcal{B}^K}\psi)(\bar{A}^K_{\alpha\beta L}\mathcal{E}^L+\bar{B}^K_{\alpha\beta L}\mathcal{B}^L)+(\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)(A^K_{\alpha\beta L}\hat{\mathcal{E}}^L+B^K_{\alpha\beta L}\hat{\mathcal{B}}^L) \\ & + (\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)(\bar{A}^K_{\alpha\beta L}\hat{\mathcal{E}}^L+\bar{B}^K_{\alpha\beta L}\hat{\mathcal{B}}^L)] \\ & + \underset{(i)}{\omega}^{\alpha\beta}[\hat{R}_{\mathcal{E}^K}(\underset{\approx}{\omega}':(A^K_{\alpha\beta L}\mathcal{E}^L+B^K_{\alpha\beta L}\mathcal{B}^L))+\hat{R}_{\mathcal{B}^K}(\underset{\approx}{\omega}':(\bar{A}^K_{\alpha\beta L}\mathcal{E}^L+\bar{B}^K_{\alpha\beta L}\mathcal{B}^L)) \\ & + \hat{R}_{\hat{\mathcal{E}}^K}(\underset{\approx}{\omega}':(A^K_{\alpha\beta L}\hat{\mathcal{E}}^L+B^K_{\alpha\beta L}\hat{\mathcal{B}}^L))+\hat{R}_{\hat{\mathcal{B}}^K}(\underset{\approx}{\omega}':(\bar{A}^K_{\alpha\beta L}\hat{\mathcal{E}}^L+\bar{B}^K_{\alpha\beta L}\hat{\mathcal{B}}^L))] \end{aligned}$$

$$(i) = 1,\ldots,6 \ . \tag{6.14}$$

Integrate (6.14) over the whole world tube (β) and by use of (6.12), we see that

$$\underset{(i)}{\omega^{\alpha\beta}} \int_{(\beta)} \{ (\tilde{\delta}_{\mathcal{E}^K}\psi)(A^K_{\alpha\beta L}\mathcal{E}^L + B^K_{\alpha\beta L}\mathcal{B}^L) + (\tilde{\delta}_{\mathcal{B}^K}\psi)(\bar{A}^K_{\alpha\beta L}\mathcal{E}^L + \bar{B}^K_{\alpha\beta L}\mathcal{B}^L)$$

$$+ (\tilde{\delta}_{\hat{\mathcal{E}}^K}\psi)(A^K_{\alpha\beta L}\hat{\mathcal{E}}^L + B^K_{\alpha\beta L}\hat{\mathcal{B}}^L) + (\tilde{\delta}_{\hat{\mathcal{B}}^K}\psi)(\bar{A}^K_{\alpha\beta L}\hat{\mathcal{E}}^L + \bar{B}^K_{\alpha\beta L}\hat{\mathcal{B}}^L)\} dV^{(4)}$$

$$= 0 \qquad\qquad (i) = 1,\ldots,6 \,. \qquad (6.15)$$

There are 6 independent integro-differential equations in (6.15). If we place restrictions on the choice of $\underset{\sim}{\phi}(\underset{\sim}{y};\underset{\sim}{\omega})$ and $\underset{\sim}{U}(\underset{\sim}{y};\underset{\sim}{\omega})$ under infinitesimal Lorentz transformation and use the admissible constitutive relations (5.19), (5.20), (5.21) and (5.22), equation (6.15) guarantees satisfaction of the parts of the zero mean conditions involving functional derivatives of ψ . The calculations in the general case are of such length and sufficiently delicate that we leave conclusive proof of this important conclusion to a further paper.

Remark. Admissible constitutive relations involve nonlinear functionals of the arguments $\mathcal{E}^K$, $\mathcal{B}^K$, $\hat{\mathcal{E}}^K$ and $\hat{\mathcal{B}}^K$, so that the Helmholtz free energy Ψ is a scalar valued function and functional of $\mathcal{E}^K$, $\mathcal{B}^K$, $\hat{\mathcal{E}}^K$ and $\hat{\mathcal{B}}^K$ not $\mathcal{E}^K_{(total)}$ and $\mathcal{B}^K_{(total)}$. Satisfaction of the zero mean conditions for $\hat{\ell}^{\alpha\beta}$, $\hat{\phi}^{\alpha\beta}$ and $\hat{\pi}^{\alpha\beta}$ then follows only if Ψ depends on both the sets of variables $\mathcal{E}^K$, $\mathcal{B}^K$ and $\hat{\mathcal{E}}^K$, $\hat{\mathcal{B}}^K$. If we take $\Psi = \psi(\mathcal{E}^K, \mathcal{B}^K)$ then the zero mean conditions are not satisfied.

The second problem is that of the general forms of ψ, $\underset{\approx}{\phi}$ and $\underset{\approx}{U}$ for which ψ and $\underset{\approx}{\phi}$ are invariant under Lorentz transformation and the components of $\underset{\approx}{U}$ have the appropriate invariant or covariant or contravariant laws of transformation under Lorentz group. We know a lot of invariant forms of ψ under Lorentz transformation (cf. R. A. Grot and A. C. Eringen [1], Y. Kazakia [23]). The problem still arises, however, of what the general invariant forms of ψ and $\underset{\approx}{\phi}$ are under Lorentz transformation when ψ and $\underset{\approx}{\phi}$ have functional arguments. We also need the specific forms for $\underset{\approx}{U}$ under Lorentz transformation.

The third problem is that of establishing representations and physical interpretations of $\underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega})$. We know that if $\underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega}) = 0$, we obtain the generalized form of Onsager's reciprocity relations from admissible constitutive relations. After we have solved the dynamical law with admissible constitutive relations that involve $\underset{\approx}{U} \neq 0$, we can compare the results with those that obtain when the generalized Onsager's reciprocity relations are satisfied. This would give unequivocal physical interpretations of $\underset{\approx}{U}(\underset{\approx}{y};\underset{\approx}{\omega}) \neq 0$. These are the lengthy calculations and are outside the scope of this thesis.

The ability of decomposing the total electromagnetic field into an external field plus an induced self field (localization residual) would appear to offer significant simplifications of certain problems in plasma dynamics.

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Résumé

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